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Dynamics in a periodic competitive model with stage structure ☆

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Abstract

In this paper, we consider a time-delayed periodic system which describes the competition among mature populations. By appealing to theories of monotone dynamical systems, periodic semiflows and uniform persistence, we analyze the evolutionary behavior of the system and establish sufficient conditions for competitive coexistence and exclusion.

Keywords: Multi-species competition; Global attractivity; Uniform persistence; Periodic coexistence state

1. Introduction

Population models with stage structure have received extensive investigations (see [2,4,5,7,10,14,18,20,22,24] and references therein). To describe a single species growth, Aiello and Freedman [1] proposed the following system:

$$\begin{aligned}\dot{x}(t) &= \alpha e^{-\gamma\tau} x(t-\tau) - \beta x^2(t),\\ \dot{y}(t) &= \alpha x(t) - \gamma y(t) - \alpha e^{-\gamma\tau} x(t-\tau),\end{aligned} \tag{1.1}$$

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where $x(t)$ and $y(t)$ denote the mature and immature populations, β and γ represent the death rates of the mature and the immature, α denotes the birth rate of the mature, and τ is the maturation age. They showed that there exists an asymptotically stable positive equilibrium, and concluded that the introduction of stage structure does not affect the permanence of the species.

In order to investigate how the stage structure affects the asymptotic behavior of the competitive species, Liu et al. [16] combined the competitive Lotka–Volterra system with system (1.1) and obtained a two-species competitive model with stage structure:

$$\begin{aligned}
\dot{x}_i(t) &= b_i e^{-d_i\tau_i} x_i(t-\tau_i) - x_i(t)\big(a_{i1}x_1(t) + a_{i2}x_2(t)\big),\\
\dot{y}_i(t) &= b_i x_i(t) - d_i y_i(t) - b_i e^{-d_i\tau_i} x_i(t-\tau_i), \quad i = 1, 2,
\end{aligned} \tag{1.2}$$

where $x_i(t)$ and $y_i(t)$ denote the mature and immature populations of the ith species, $a_{ij} > 0$, b_i and d_i denote the birth rate of the ith mature population and the death rate of the ith immature population, respectively, τ_i is the maturation age of species i. One of the basic assumptions is that the immature does not compete with the other species. Since the equations for mature populations are decoupled from those for immature populations, it suffices to study the global dynamics of subsystem (1.2). The authors of [16] defined $\xi_i = d_i\tau_i$ as the degree of stage, and concluded that if

$$\frac{a_{12}}{a_{22}} < \frac{b_1 e^{-\xi_1}}{b_2 e^{-\xi_2}} < \frac{a_{11}}{a_{21}},$$

then system (1.2) is permanent. Furthermore, Liu et al. generalized the above system to an autonomous competitive system for n species in [15] and a T-periodic competitive system for n species in [17]:

$$\begin{aligned}
\dot{x}_i(t) &= B_i(t)x_i(t-\tau_i) - x_i(t)\sum_{j=1}^{n} a_{ij}(t)x_j(t),\\
\dot{y}_i(t) &= b_i(t)x_i(t) - d_i(t)y_i(t) - B_i(t)x_i(t-\tau_i), \quad 1 \leqslant i \leqslant n,
\end{aligned} \tag{1.3}$$

where $b_i(t)$, $a_{ii}(t)$, $d_i(t) > 0$, $a_{ij}(t) \geqslant 0$, and

$$B_i(t) = b_i(t-\tau_i)e^{-\int_{t-\tau_i}^{t} d_i(s)\,ds}, \quad 1 \leqslant i \leqslant n.$$

They obtained that if

$$B_i^l > \sum_{j\neq i} a_{ij}^m B_j^m / a_{jj}^l, \quad 1 \leqslant i \leqslant n, \tag{1.4}$$

then system (1.3) is permanent, where

$$a_{ij}^l = \inf_t a_{ij}, \quad a_{ij}^m = \sup_t a_{ij}, \quad B_j^l = \inf_t B_j, \quad B_j^m = \sup_t B_j, \quad 1 \leqslant i, j \leqslant n.$$

It is easy to see that condition (1.4) is very strong. There should exist more natural conditions in terms of average integrals of certain functions over the interval $[0, T]$. Also motivated by systems (1.1)–(1.3), we consider the following periodic system of competing mature populations

$$\begin{aligned}
\dot{u}_i(t) &= u_i(t-\tau_i)F_i\big(t, u_i(t-\tau_i)\big) - u_i(t)G_i\big(t, u_1(t), \ldots, u_m(t)\big)\\
&= f_i\big(t, u_1(t), \ldots, u_m(t), u_i(t-\tau_i)\big), \quad 1 \leqslant i \leqslant m,
\end{aligned} \tag{1.5}$$

where the continuous function $f_i(t, u_1, \dots, u_m, v_i)$ is T-periodic in t, and Lipschitzian in $(u_1, \dots, u_m, v_i)$ in any bounded subset of $\mathbb{R}_+^{m+1}$, $i = 1, 2, \dots, m$. For this model, we have the same basic assumption: the immature of a species does not compete with other species. Note that system (1.5) is also a general form of Ayala's system (see, e.g., [3,13] for the autonomous case, and [6] for the nonautonomous case).

The purpose of this paper is to analyze the global dynamics of system (1.5). By appealing to the theory of periodic semiflows, we established sufficient conditions for the existence of periodic coexistence state, global persistence and extinction in terms of spectral radii of the Poincaré maps associated with linear periodic delay equations. In the case where the delays are integer multiples of the period, these conditions can be determined by the average integrals along certain periodic functions. When applied to system (1.3), the obtained conditions are necessary to those in [17], and the results improve those obtained in [17].

The organization of this paper is as follows. In Section 2, we give some preliminary results on the spectral radius of the Poincaré map associated with a linear periodic and delayed equation, and threshold dynamics in a scalar periodic and delayed system. In Section 3, we first analyze the global dynamics in two-species competitive system (1.5) by using the theory of competitive systems on Banach spaces [12]. In Section 4, we first investigate the uniform persistence of multi-species competitive systems by two-side comparison method, and then obtain natural invasibility conditions for the persistence and coexistence states of three-species competitive systems by using the theory of uniform persistence.

2. Preliminaries

In this section, we first introduce some basic notations, and then present the preliminary results on scalar delay differential equations. Let τ, τ_1 and τ_2 be positive numbers, and

$$Y = C\big([-\tau, 0], \mathbb{R}\big), \qquad Y^+ = C\big([-\tau, 0], \mathbb{R}_+\big), \qquad X_i = C\big([-\tau_i, 0], \mathbb{R}\big),$$
$$X_i^+ = C\big([-\tau_i, 0], \mathbb{R}_+\big), \quad i = 1, 2, \qquad X = X_1 \times X_2, \qquad X^+ = X_1^+ \times X_2^+.$$

Then (Y, Y^+), (X_i, X_i^+) and (X, X^+) are ordered Banach spaces. For $\varphi, \psi \in Y$, we write

$$\begin{aligned} &\varphi \leqslant \psi \quad \text{if } \psi - \varphi \in Y^+,\\ &\varphi < \psi \quad \text{if } \psi - \varphi \in Y^+ \setminus \{0\},\\ &\varphi \ll \psi \quad \text{if } \psi - \varphi \in \operatorname{int}(Y^+). \end{aligned}$$

For $\varphi, \psi \in X_1$, X_2, X and $\mathbb{R}_+^2$, we have the same notations for the partial orders. Let $K = X_1^+ \times (-X_2^+)$. Then (X, K) is also an ordered Banach space. In a similar way, we can define $\leqslant_K$, $<_K$, $\ll_K$. By an order interval $[\varphi, \psi]_K$ on X, we mean that

$$[\varphi, \psi]_K = \{\xi \in X\colon \varphi \leqslant_K \xi \leqslant_K \psi\}.$$

For a linear operator P, we denote the spectral radius of P by $r(P)$.

Consider a linear scalar equation with delay τ

$$\dot{u} = a(t)u(t) + b(t)u(t - \tau). \tag{2.1}$$

Assume that

(H) $a(t)$ and $b(t)$ are T-periodic and continuous, and $b(t) > 0$ $\forall t \geqslant 0$.

Let $u_t(\varphi)$ be the solution semiflow for Eq. (2.1) defined by $u_t(\varphi)(s) = u(t+s, \varphi)$ $\forall s \in [-\tau, 0]$, where $u(t, \varphi)$ is the unique solution of (2.1) with $u(s, \varphi) = \varphi(s) \in Y^+$. In what follows, we always apply $u(t, \varphi)$ to a solution of a certain system, and $u_t(\varphi)$ to the associated solution semiflow. Since $b(t) > 0$, it follows that $u_t(\varphi) \geqslant 0$ $\forall \varphi \in Y^+$, $t \geqslant 0$. Define the Poincaré map $P : Y^+ \to Y^+$ by $P(\varphi) = u_T(\varphi)$. Then, $P^n(\varphi) = u_{nT}(\varphi)$ for integer $n \geqslant 0$.

The following result associates the spectral radius $r(P)$ with an integral of the coefficients of Eq. (2.1).

Proposition 2.1. *$r = r(P)$ is positive and is an eigenvalue of P with a positive eigenfunction φ^*. Moreover, if $\tau = kT$ for some integer $k \geqslant 0$, then $r - 1$ has the same sign as $\int_0^T (a(t) + b(t))\,dt$.*

Proof. By assumption (H), [8, Theorem 3.6.1] and [19, Lemma 5.3.2], there exists an integer m, $mT \geqslant 2\tau$, such that P^m is compact and strongly positive. By the Krein–Rutman theorem (see, e.g., [11, Theorem 7.2]), $r_m = r(P^m) > 0$ and is an algebraically simple eigenvalue of P^m with an eigenfunction $\varphi_m^* \gg 0$. Since P is a bounded linear operator on Y^+, $r_m = r^m$. Moreover, the spectrum of P consists of the point spectrum of P and the possible accumulation point being zero (see, e.g., [8, p. 192]). Thus, r is a positive eigenvalue of P. Let $P\varphi^* = r\varphi^*$. Without lose of generality, we assume $\varphi^*(s_0) > 0$ for some $s_0 \in [-\tau, 0]$. Since $P^m\varphi^* = r^m\varphi^* = r_m\varphi^*$, we have $\varphi^* = c\varphi_m^*$ for some positive constant c. Thus $\varphi^* \gg 0$.

Letting $u(t) = e^{\lambda t} v(t)$, we obtain a linear periodic equation with parameter λ,

$$\dot v(t) = \big(a(t) - \lambda\big)v(t) + b(t)e^{-\lambda\tau} v(t-\tau). \tag{2.2}$$

Define $Q : Y^+ \to Y^+$ by $Q(\varphi) = v_T(\varphi)$, where $v_t(\varphi)$ is the solution semiflow of Eq. (2.2). Let E_λ be a map from Y^+ to Y^+ defined by $[E_\lambda(\varphi)](s) = e^{\lambda s}\varphi(s)$ $\forall s \in [-\tau, 0]$. Then

$$Q(\varphi)(s) = v_T(\varphi)(s) = v(T+s, \varphi) = e^{-\lambda(T+s)} u\big(T+s, E_\lambda(\varphi)\big) \quad \forall s \in [-\tau, 0],$$

and hence,

$$Q(\varphi) = e^{-\lambda T} E_{-\lambda}\big(u_T\big(E_\lambda(\varphi)\big)\big) = e^{-\lambda T} E_{-\lambda}\big(P\big(E_\lambda(\varphi)\big)\big).$$

Thus, $Q(E_{-\lambda}(\varphi^*)) = e^{-\lambda T} E_{-\lambda}(P(\varphi^*)) = re^{-\lambda T} E_{-\lambda}(\varphi^*)$. Let $\lambda_0 = (1/T)\ln r$. Then $E_{-\lambda_0}\varphi^*$ is a positive fixed point of Q. Thus $v_0(t) = v(t, E_{-\lambda_0}(\varphi^*))$ is a positive T-periodic solution of (2.2), and $u(t) = v_0(t)e^{\lambda_0 t} > 0$ for $t \geqslant -\tau$. In particular, if $\tau = kT$ for some integer $k \geqslant 0$, then $v_0(t)$ satisfies

$$\frac{\dot v_0(t)}{v_0(t)} = a(t) - \lambda_0 + b(t)e^{-\lambda_0\tau} \quad \forall t \geqslant 0.$$

Integrating both sides of the above equation from 0 to T, we get

$$\lambda_0 = \frac{1}{T}\int_0^T \big(a(t) + e^{-\lambda_0\tau} b(t)\big)\,dt.$$

Note that

$$G(\lambda) = \frac{1}{T}\int_0^T a(t)\,dt + \frac{1}{T}e^{-\lambda\tau}\int_0^T b(t)\,dt$$

is strictly decreasing, and λ_0 is the unique solution of $\lambda = G(\lambda)$, we have $\lambda_0 G(0) > 0$, i.e., $(r-1)\int_0^T (a(t)+b(t))\,dt > 0$. The desired results are established. $\Box$

Let us consider a nonlinear T-periodic equation

$$\begin{cases} \dot{u} = f\big(t, u(t), u(t-\tau)\big), \\ u(s) = \varphi(s), \quad -\tau \leqslant s \leqslant 0, \end{cases} \tag{2.3}$$

where $\varphi \in Y^+$ is an initial function to be specified later.

Assume that the continuous function $f(t, v_1, v_2)$ is T-periodic in t and Lipschitzian in (v_1, v_2) in any bounded subset of $\mathbb{R}^2_+$, and satisfies

(C1) $f(t,0,0) = 0$, $f(t,0,v_2) \geqslant 0$, $(\partial/\partial v_2) f(t, v_1, v_2) > 0$ $\forall v_1, v_2 \geqslant 0$;
(C2) f is strictly sublinear, i.e., for any $\alpha \in (0,1)$, $f(t, \alpha v_1, \alpha v_2) > \alpha f(t, v_1, v_2)$ $\forall v_1, v_2 \geqslant 0$;
(C3) there exists a positive number $L > 0$ such that $f(t, L, L) \leqslant 0$.

Let P_u be the Poincaré map of the linearized equation associated with Eq. (2.3) at $u \equiv 0$, and $r = r(P_u)$. Then we have the following threshold type result on the global dynamics of (2.3).

Theorem 2.1. *Let* (C1)–(C3) *hold. Then the following statements hold.*

(i) *If $r \leqslant 1$, then zero solution is globally asymptotically stable for Eq.* (2.3) *with respect to Y^+.*
(ii) *If $r > 1$, then Eq.* (2.3) *has a unique positive T-periodic solution $u(t, \varphi_0)$, and $u(t, \varphi_0)$ is globally asymptotically stable with respect to $Y^+ \setminus \{0\}$.*

Proof. Let $a(t) = (\partial/\partial v_1) f(t,0,0)$, $b(t) = (\partial/\partial v_2) f(t,0,0)$. Since f is strictly sublinear, $f(t, v_1, v_2) \leqslant a(t)v_1 + b(t)v_2$. Note that $b(t) > 0$, $f(t,0,v_2) \geqslant 0$. By the comparison theorem [19, Theorem 5.1.1] and the positivity theorem [19, Theorem 5.2.1], each solution $u(t,\varphi)$ of Eq. (2.3) with initial value $\varphi \in Y^+$ exists globally, and $u(t,\varphi) \geqslant 0$ $\forall t \geqslant -\tau$. Since $(\partial/\partial v_2) f(t, v_1, v_2) > 0$, the nonautonomous version of [19, Theorem 5.3.4] implies that for any $\varphi, \psi \in Y^+$ with $\varphi \leqslant \psi$, $u_t(\varphi) \leqslant u_t(\psi)$ $\forall t \geqslant 0$; and if $\varphi < \psi$, then $u_t(\varphi) \ll u_t(\psi)$ $\forall t \geqslant 2\tau$. Define $S_u : Y^+ \to Y^+$ by $S_u(\varphi) = u_T(\varphi)$. Then S_u is monotone, and S_u^n is strongly monotone for $nT \geqslant 2\tau$. Moreover, the strict sublinearity of f implies that S_u is strictly sublinear (see the proof of [25, Theorem 3.3]).

By the continuity and differentiability of solutions with respect to initial values, it follows that the Poincaré map S_u is differentiable at zero, and $DS_u(0) = P_u$. Since $b(t) > 0$, as in the proof of Proposition 2.1, $(DS_u(0))^n$ is compact and strongly positive for all $nT \geqslant 2\tau$.

Let us consider $S_u^{n_0}$, where $n_0 T \geqslant 2\tau$. Then, $S_u^{n_0}$ is strongly monotone, and $(DS_u(0))^{n_0}$ is compact and strongly positive.

For any $\beta \geqslant 1$, since f is strictly sublinear, we have $f(t, \beta L, \beta L) < \beta f(t, L, L) \leqslant 0$. Thus, [19, Remark 5.2.1] implies that for any $\beta \geqslant 1$, the order interval $V_\beta = [0, \beta L]$ is a positive invariant set for S_u, where $V_\beta = \{\varphi \in Y^+ \colon\ 0 \leqslant \varphi(s) \leqslant \beta L, s \in [-\tau, 0]\}$. By [8, Theorem 3.6.1], $S_u^{n_0} \colon V_\beta \to V_\beta$ is compact for any fixed $\beta \geqslant 1$. Then the closure of $S_u^{n_0}([\varphi, \psi])$ is a compact subset of V_β for any $\varphi, \psi \in V_\beta$ with $\varphi \leqslant \psi$. Furthermore, $DS_u^{n_0}(0) = (DS_u(0))^{n_0}$, which is compact and strongly positive. Note that S_u is strictly sublinear, $S_u^{n_0}$ is strongly monotone, and equalities $r\{(DS_u(0))^{n_0}\} = [r(DS_u(0))]^{n_0} = (r(P_u))^{n_0} = r^{n_0}$ hold. By [26, Theorem 2.3], as applied to $S_u^{n_0}$, we have the following conclusions.

(i) If $r \leqslant 1$, then zero is a globally asymptotically stable fixed point of $S_u^{n_0}$ with respect to V_β.
(ii) If $r > 1$, then $S_u^{n_0}$ has a unique positive fixed point φ_0 in V_β, and φ_0 is globally asymptotically stable with respect to $V_\beta \setminus \{0\}$.

By the arbitrariness of β, the above results hold on the whole space Y^+ for $S_u^{n_0}$. It then follows that zero solution of Eq. (2.3) is globally asymptotically stable in case (i); and Eq. (2.3) admits the unique, positive and n_0T-periodic solution $u(t, \varphi_0)$ in case (ii). It remains to prove that $u(t, \varphi_0)$ is T-periodic. By Proposition 2.1, we know that there exists a positive eigenfunction φ^* such that $DS_u(0)(\varphi^*) = r\varphi^*$. In the case of $r > 1$, for any small $\varepsilon > 0$, it is easy to find an increasing sequence $0 \ll \varepsilon\varphi^* \ll S_u(\varepsilon\varphi^*) \leqslant S_u^2(\varepsilon\varphi^*) \leqslant \cdots \leqslant S_u^n(\varepsilon\varphi^*) \leqslant \cdots$ (see the proof of [29, Theorem 2.1]). On the other hand, $S_u^{n_0 n}(\varepsilon\varphi^*) \to \varphi_0$ as $n \to \infty$. Thus, by the monotonicity of the sequence of $S_u^n(\varepsilon\varphi^*)$ and the continuity of S_u, φ_0 is a fixed point of S_u. That is, $u(t, \varphi_0)$ is a T-periodic solution. □

3. Two-species competition

In this section, we use the theory of competitive systems on Banach spaces (see [12]) to analyze the global dynamics of system (1.5) in the case of two-species competition.

In the case of $m = 2$, we assume that periodic system (1.5) satisfies:

(H1) $F_i(t, u_i) > 0$, $(\partial/\partial u_i)(u_i F_i(t, u_i)) > 0$, and $(\partial/\partial u_j) G_i(t, u_1, u_2) > 0$ for $t \geqslant 0$, $u_i \geqslant 0$, $1 \leqslant i \neq j \leqslant 2$.
(H2) $f_1(t, \cdot, 0, \cdot)$ and $f_2(t, 0, \cdot, \cdot)$ are strictly sublinear on $\mathbb{R}_+^2$, and $f_1(t, L, 0, L) \leqslant 0$ and $f_2(t, 0, L, L) \leqslant 0$ for some number $L > 0$.

Consider the linearization of system (1.5) at zero:

$$\dot{u}_1(t) = b_1(t) u_1(t - \tau_1) - a_1(t) u_1(t), \tag{3.1}$$

$$\dot{u}_2(t) = b_2(t) u_2(t - \tau_2) - a_2(t) u_2(t), \tag{3.2}$$

where $b_i(t) = F_i(t, 0)$, $a_i(t) = G_i(t, 0, 0)$. Let $P_1^{(0)}$ and $P_2^{(0)}$ be the Poincaré maps associated with Eqs. (3.1) and (3.2), $r_{01} = r(P_1^{(0)})$ and $r_{02} = r(P_2^{(0)})$ be the spectral radii of $P_1^{(0)}$

and $P_2^{(0)}$, respectively. Assume that

(H3) $r_{01} > 1, r_{02} > 1$.

By Theorem 2.1, it then follows that there exists a unique positive T-periodic solution $u^{(1)}(t)$ to

$$\begin{aligned}\dot{u}_1(t) &= u_1(t-\tau_1)F_1\big(t, u_1(t-\tau_1)\big) - u_1(t)G_1\big(t, u_1(t), 0\big)\\ &= f_1\big(t, u_1(t), 0, u_1(t-\tau_1)\big),\end{aligned} \tag{3.3}$$

and $u^{(1)}(t)$ is globally asymptotically stable with respect to $X_1^+ \setminus \{0\}$. The similar results hold for the equation

$$\begin{aligned}\dot{u}_2(t) &= u_2(t-\tau_2)F_2\big(t, u_2(t-\tau_2)\big) - u_2(t)G_2\big(t, 0, u_2(t)\big)\\ &= f_2\big(t, 0, u_2(t), u_2(t-\tau_2)\big).\end{aligned} \tag{3.4}$$

Let $u^{(2)}(t)$ be the unique positive T-periodic solution to Eq. (3.4).

Obviously, $(u^{(1)}(t), 0)$ and $(0, u^{(2)}(t))$ are T-periodic solutions of system (1.5). Linearizing system (1.5) at $(u^{(1)}(t), 0)$, we have

$$\dot{u}_1(t) = b_1^{(1)}(t)u_1(t-\tau_1) - a_{11}^{(1)}(t)u_1(t) - a_{12}^{(1)}(t)u_2(t), \tag{3.5}$$

$$\dot{u}_2(t) = b_2^{(1)}(t)u_2(t-\tau_2) - a_{22}^{(1)}(t)u_2(t), \tag{3.6}$$

where

$$\begin{aligned}
b_1^{(1)}(t) &= u^{(1)}(t-\tau_1)\frac{\partial}{\partial u_1}F_1\big(t, u^{(1)}(t-\tau_1)\big) + F_1\big(t, u^{(1)}(t-\tau_1)\big),\\
b_2^{(1)}(t) &= F_2(t, 0),\\
a_{11}^{(1)}(t) &= G_1\big(t, u^{(1)}(t), 0\big) + u^{(1)}(t)\frac{\partial}{\partial u_1}G_1\big(t, u^{(1)}(t), 0\big),\\
a_{12}^{(1)}(t) &= u^{(1)}(t)\frac{\partial}{\partial u_2}G_1\big(t, u^{(1)}(t), 0\big),\\
a_{22}^{(1)}(t) &= G_2\big(t, u^{(1)}(t), 0\big).
\end{aligned}$$

Similarly, we have the linearized system of system (1.5) at $(0, u^{(2)}(t))$:

$$\dot{u}_1(t) = b_1^{(2)}(t)u_1(t-\tau_1) - a_{11}^{(2)}(t)u_1(t), \tag{3.7}$$

$$\dot{u}_2(t) = b_2^{(2)}(t)u_2(t-\tau_2) - a_{21}^{(2)}(t)u_1(t) - a_{22}^{(2)}(t)u_2(t), \tag{3.8}$$

where

$$\begin{aligned}
b_1^{(2)}(t) &= F_1(t, 0),\\
b_2^{(2)}(t) &= u^{(2)}(t-\tau_2)\frac{\partial}{\partial u_2}F_2\big(t, u^{(2)}(t-\tau_2)\big) + F_2\big(t, u^{(2)}(t-\tau_2)\big),\\
a_{11}^{(2)}(t) &= G_1\big(t, 0, u^{(2)}(t)\big),\\
a_{21}^{(2)}(t) &= u^{(2)}(t)\frac{\partial}{\partial u_1}G_2\big(t, 0, u^{(2)}(t)\big),
\end{aligned}$$

$$a_{22}^{(2)}(t) = G_2\big(t, 0, u^{(2)}(t)\big) + u^{(2)}(t)\frac{\partial}{\partial u_2}G_2\big(t, 0, u^{(2)}(t)\big).$$

Let $P_2^{(1)}$ and $P_1^{(2)}$ be the Poincaré maps of Eqs. (3.6) and (3.7), respectively, and denote their spectral radii by $r_{12} = r(P_2^{(1)})$, $r_{21} = r(P_1^{(2)})$. Let $\varphi^*(s_1) = u^{(1)}(s_1)\ \forall s_1 \in [-\tau_1, 0]$, $\varphi^{**}(s_2) = u^{(2)}(s_2)\ \forall s_2 \in [-\tau_2, 0]$, and set $E_0 = (0, 0)$, $E_1 = (\varphi^*, 0)$, $E_2 = (0, \varphi^{**})$. For any $\psi \in X^+$, denote by $u(t, \psi)$ the solution of system (1.5). Let $u_t(\psi)$ be the solution semiflow associated with system (1.5). For convenience, we set $X^0 = \{(\psi_1, \psi_2) \in X^+\colon \psi_i \neq 0,\ i = 1, 2\}$. Then we have the following result.

Theorem 3.1. *Let* (H1)–(H3) *hold and suppose that* $r_{12} > 1$, $r_{21} > 1$. *Then for system* (1.5):

(i) *System* (1.5) *has two positive* T*-periodic solutions* $u(t, \phi^*)$ *and* $u(t, \phi^{**})$ *satisfying* $u(t, \phi^{**}) \leqslant_K u(t, \phi^*)$, $t \geqslant 0$, *where* $\phi^*, \phi^{**} \in \operatorname{int}(X^+)$ *with* $\phi^{**} \leqslant_K \phi^*$.

(ii) *Equality*

$$\lim_{t\to\infty} \big\| u(t, \psi) - u(t, \phi^*)\big\| = 0$$

holds for every $\psi = (\psi_1, \psi_2) \in X^+$ *with* $\phi^* \leqslant_K \psi <_K E_1$ *and* $\psi_2 \neq 0$. *Symmetrically,*

$$\lim_{t\to\infty} \big\| u(t, \psi) - u(t, \phi^{**})\big\| = 0$$

for every $\psi = (\psi_1, \psi_2) \in X^+$ *with* $E_2 <_K \psi \leqslant_K \phi^{**}$ *and* $\psi_1 \neq 0$.

(iii) *Equality*

$$\lim_{t\to\infty} \operatorname{dist}\big(u(t, \psi), \big[u(t, \phi^{**}), u(t, \phi^*)\big]_K\big) = 0$$

holds for any point $\psi \in X^0$.

In particular, in the case where $\tau_i = k_i T$ *for some integers* k_i, $i = 1, 2$, *if assumptions* (H1) *and* (H2) *hold, and*

$$\int_0^T \big(b_i(t) - a_i(t)\big)\,dt > 0, \qquad \int_0^T \big(b_j^{(i)}(t) - a_{jj}^{(i)}(t)\big)\,dt > 0$$

for $i \neq j$ *and* $i, j = 1, 2$, *then the above results hold.*

In order to prove Theorem 3.1, we need the following two lemmas. In the rest of this section, we always use S to denote the Poincaré map associated with system (1.5).

Lemma 3.1. *The Poincaré map* $S : X^+ \to X^+$ *is strictly monotone with respect to* $\leqslant_K$, *and is a bounded map.*

Proof. For any $\psi \in X^+$, by the positivity theorem [19, Theorem 5.2.1] and assumption (H1), the solution $u(t, \psi)$ of system (1.5) is nonnegative on its existence interval. Note that assumption (H1) implies the inequalities

$$f_1(t, u_1, u_2, v_1) \leqslant f_1(t, u_1, 0, v_1) \quad \text{and} \quad f_2(t, u_1, u_2, v_2) \leqslant f_2(t, 0, u_2, v_2)$$

for $u_i, v_i \geqslant 0$, $i = 1, 2$. Since the solutions for Eqs. (3.3) and (3.4) exist globally, by the comparison theorem [19, Theorem 5.1.1], the solution $u(t, \psi)$ for system (1.5) globally exists for any $\psi \in X^+$. By assumption (H1), it easily follows that the solution $u_1(t, \varphi_1)$ of Eq. (3.3) is bounded by $B = \max\{L, \|\varphi_1\|\}$, and hence solutions for Eq. (3.3) are uniformly bounded. The same conclusions hold for Eq. (3.4). Therefore, solutions for system (1.5) are also uniformly bounded.

Let $u_t(\psi)$ be the solution semiflow of system (1.5) with $u_0(\psi) = \psi \in X^+$. Then, $u_t(\psi) \geqslant 0$ for all $t \geqslant 0$. Moreover, if $\varphi, \psi \in X^+$ with $\varphi \leqslant_K \psi$, by the comparison theorem and the transformation $U_1 = u_1, U_2 = -u_2$, it easily follows that $u_t(\varphi) \leqslant_K u_t(\psi)$ for all $t \geqslant 0$. Let $S : X^+ \to X^+$ be the Poincaré map associated with system (1.5), i.e., $S(\cdot) = u_T(\cdot)$. Then S is monotone with respect to $\leqslant_K$, and S is a bounded map.

It remains to prove that S is strictly monotone with respect to $\leqslant_K$, i.e., $S(\varphi) <_K S(\psi)$ if $\varphi <_K \psi$. Suppose, by contradiction, that $S(\varphi) = S(\psi)$. Let $u(t, \varphi) = (u_1(t, \varphi), u_2(t, \varphi))$, $u(t, \psi) = (u_1(t, \psi), u_2(t, \psi))$. Then $u_i(t_i, \varphi) = u_i(t_i, \psi)$ for all $t_i \in [T - \tau_i, T], i = 1, 2$. Thus,

$$\begin{aligned} 0 &= \dot{u}_i(t_i, \varphi) - \dot{u}_i(t_i, \psi) \\ &= u_i(t_i - \tau_i, \varphi) F_i\big(t_i, u_i(t_i - \tau_i, \varphi)\big) - u_i(t_i - \tau_i, \psi) F_i\big(t_i, u_i(t_i - \tau_i, \psi)\big) \end{aligned}$$

for $t_i \in (T - \tau_i, T]$. Since $u_i F_i(t, u_i)$ is strictly increasing, $u_i(t_i - \tau_i, \varphi) = u_i(t_i - \tau_i, \psi)$. Therefore, $u_i(t_i, \varphi) = u_i(t_i, \psi)$ for $t_i \in (T - 2\tau_i, T], i = 1, 2$. By induction, we have $u_i(t_i, \varphi) = u_i(t_i, \psi)$ for $t_i \in [-\tau_i, 0]$, i.e., $\varphi = \psi$, which contradicts to $\varphi <_K \psi$. Thus we have $S(\varphi) <_K S(\psi)$. $\square$

Lemma 3.2. *Suppose $u^*(t) = (u_1^*(t), u_2^*(t))$ is a T-periodic solution of Eq.* (1.5) *with $u_i^*(t) \geqslant 0$ for some $1 \leqslant i \leqslant 2$, and $u_j^*(t) \equiv 0$ for $j \neq i$. Let P_j be the Poincaré map of*

$$\dot{u}_j(t) = F_j(t, 0) u_j(t - \tau_j) - G_j\big(t, u_1^*(t), u_2^*(t)\big) u_j(t).$$

If $r_j = r(P_j) > 1$, then for any integer $n_0 \geqslant 1$, there exists $\delta > 0$ such that

$$\limsup_{n \to \infty} \big\| S^{n_0 n}(\psi) - \psi^* \big\| \geqslant \delta \quad \textit{for all } \psi \in \operatorname{int}(X^+),$$

where $\psi^* \in X^+$ is the initial function of $u^*(t)$.

Proof. Since $u^*(t)$ is also an $n_0 T$-periodic solution of $n_0 T$-periodic system (1.5), and $r\{(P_j)^{n_0}\} = [r(P_j)]^{n_0} = r_j^{n_0} > 1$, without loss of generality, we can assume that $n_0 = 1$.

It suffices to prove that there exists $\delta > 0$ such that for any $\psi \in \operatorname{int}(X^+)$ with $\|\psi - \psi^*\| < \delta$, there exists $N \geqslant 1$ such that $\|S^N(\psi) - \psi^*\| \geqslant \delta$. Let $b_1 = \min_{t \in [0,T]} F_j(t, 0)$. For any $\varepsilon \in (0, b_1)$, let r^ε be the spectral radius of the Poincaré map associated with

$$\dot{u}(t) = \big(F_j(t, 0) - \varepsilon\big) u(t - \tau_j) - \Big(G_j\big(t, u_1^*(t), u_2^*(t)\big) + \varepsilon\Big) u(t). \tag{3.9}$$

Then $\lim_{\varepsilon \to 0} r^\varepsilon = r_j > 1$. In what follows, we fix a sufficient small $\varepsilon \in (0, b_1)$ such that $r^\varepsilon > 1$. For this fixed ε, assumption (H1) implies that there exists $\delta_1 > 0$ such that

$$F_j(t, u_j) > F_j(t, 0) - \varepsilon \quad \forall t \in [0, \infty)\ \forall u_j \in [0, \delta_1).$$

Let $b_2 = \max_{t\in[0,T]} \|u^*(t)\|$. By the uniform continuity of G_j on the set $[0,\infty) \times [0, b_2+1]^2$, there exists $\delta_2 > 0$ such that

$$\left|G_j(t,u_1,u_2) - G_j\left(t,u_1',u_2'\right)\right| < \varepsilon \quad \forall t \in [0,\infty),$$

for any $u = (u_1,u_2), u' = (u_1',u_2') \in [0, b_2+1]^2$ with $\|u-u'\| < \delta_2$. By the continuous dependence of solutions on initial values, there exists $\delta > 0$ such that for any $\psi \in \mathrm{int}(X^+)$ with $\|\psi - \psi^*\| < \delta$,

$$\left\|u(t,\psi) - u^*(t)\right\| < \delta' = \min(1,\delta_1,\delta_2) \quad \forall t \in [0,T).$$

Proceeding by contradiction, assume that there exists $\bar\psi = (\bar\psi_1,\bar\psi_2) \in \mathrm{int}(X^+)$ with $\|\bar\psi - \psi^*\| < \delta$ such that $\|S^n(\bar\psi) - \psi^*\| < \delta$ for all $n \geqslant 1$. For any $t \geqslant 0$, let $t = nT + t'$, where $t' \in [0,T)$, $n = [t/T]$ is the greatest integer less than or equal to t/T. Then,

$$\left\|u(t,\bar\psi) - u^*(t)\right\| = \left\|u\left(t', S^n(\bar\psi)\right) - u^*(t')\right\| < \delta' \quad \forall t \geqslant 0.$$

Let $u(t,\bar\psi) = (\bar u_1(t), \bar u_2(t))$. Then

$$F_j\left(t, \bar u_j(t)\right) > F_j(t,0) - \varepsilon,$$

and

$$\left|G_j\left(t,\bar u_1(t),\bar u_2(t)\right) - G_j\left(t,u_1^*(t),u_2^*(t)\right)\right| < \varepsilon \quad \forall t \geqslant 0.$$

Thus,

$$\begin{aligned}\dot{\bar u}_j(t) &= \bar u_j(t-\tau_j)F_j\left(t,\bar u_j(t-\tau_j)\right) - \bar u_j(t)G_j\left(t,\bar u_1(t),\bar u_2(t)\right)\\ &> \left(F_j(t,0) - \varepsilon\right)\bar u_j(t-\tau_j) - \bar u_j(t)\left(G_i\left(t,u_1^*,u_2^*\right) + \varepsilon\right) \quad \forall t \geqslant 0.\end{aligned} \tag{3.10}$$

As in the proof of Proposition 2.1, Eq. (3.9) has a solution $u^0(t) = v_0(t)e^{\lambda_0 t}$, where $v_0(t)$ is a positive, T-periodic and continuous function, $\lambda_0 = (1/T)\ln r^\varepsilon > 0$. Let $\varphi_0(s) = u^0(s)$, $s \in [-\tau_j, 0]$. Then $\varphi_0 \gg 0$. Since $\bar\psi_j \gg 0$, there exists $\eta > 0$ such that $\eta\varphi_0 \leqslant \bar\psi_j$. By the comparison theorem and inequality (3.10), we have $\bar u_j(t) \geqslant u_j^\varepsilon(t,\bar\psi_j) \geqslant \eta u^0(t)$, where $u_j^\varepsilon(t,\bar\psi_j)$ is the solution of (3.9) with $u_j^\varepsilon(s,\bar\psi_j) = \bar\psi_j(s)$ $\forall s \in [-\tau_j, 0]$. Therefore,

$$\lim_{t\to\infty} \bar u_j(t) \geqslant \lim_{t\to\infty} \eta u^0(t) = \infty.$$

Thus $S^n(\bar\psi)$ is unbounded, a contradiction. □

Proof of Theorem 3.1. Note that the Poincaré map $S: X^+ \to X^+$ is α-condensing and S^n is compact for sufficiently large n (see, e.g., [8, Theorem 3.6.1]). We then proceed with two steps. The first step is to verify the basic assumptions in [12] (see also [28, Section 2.4]) for competitive systems on Banach spaces, and apply a compression theorem [28, Theorem 2.4.2] to S^{n_0}, where n_0 is an appropriate positive integer. In the second step, we prove that fixed points ϕ^* and ϕ^{**} of S^{n_0} are actually fixed points of S.

Step 1. So far, we have shown that (1) $u^{(1)}(t)$ and $u^{(2)}(t)$ are stable positive T-periodic solutions for Eqs. (3.3) and (3.4), respectively, and they attract all of the solutions except for the trivial solution; (2) the Poincaré map S for system (1.5) is bounded and strictly monotone with respect to $\leqslant_K$ (see Lemma 3.1).

Let S_{u_1} and S_{u_2} be the Poincaré maps of Eqs. (3.3) and (3.4), respectively. Since $X_1^+ \times \{0\}$ and $\{0\} \times X_2^+$ are clearly invariant sets for system (1.5), we have $S = (S_{u_1}, 0)$ on $X_1^+ \times \{0\}$, $S = (0, S_{u_2})$ on $\{0\} \times X_1^+$. Therefore,

$$\lim_{n\to\infty} S^n\big((\varphi_1, 0)\big) = E_1 \quad \text{for any } \varphi_1 \in X_1^+ \setminus \{0\}, \quad \text{and}$$
$$\lim_{n\to\infty} S^n\big((0, \varphi_2)\big) = E_2 \quad \text{for any } \varphi_2 \in X_2^+ \setminus \{0\}.$$

We claim the following.

Claim. *For any $\varphi = (\varphi_1, \varphi_2) \in X^0$, $u(t, \varphi) \gg 0$ for $t \geqslant \tau = \max(\tau_1, \tau_2)$. In particular, $S^n(\varphi) \gg 0$ for all $nT \geqslant 2\tau$.*

Indeed, for each $i = 1, 2$, we assume that $\varphi_i(\theta_i) > 0$ for some $\theta_i \in [-\tau_i, 0]$, $i = 1, 2$. Then $u_i(\tau_i + \theta_i, \varphi) > 0$. In fact, if $u_i(\tau_i + \theta_i, \varphi) = 0$, then

$$\dot u_i(\tau_i + \theta_i, \varphi) = u_i(\theta_i, \varphi) F_i\big(\tau_i + \theta_i, u_i(\theta_i, \varphi)\big) = \varphi_i(\theta_i) F_i\big(\tau_i + \theta_i, \varphi_i(\theta_i)\big) > 0,$$

which implies that $u_i(t_i', \varphi) < 0$ for some $t_i' < \tau_i + \theta_i$. However, by the proof of Lemma 3.1, $u_i(t, \varphi) \geqslant 0$ for all $t \geqslant -\tau_i$, a contradiction. Thus, we have $u_i(\tau_i + \theta_i, \varphi) > 0$. On the other hand,

$$\begin{aligned}\dot u_i(t, \varphi) &= u_i(t - \tau_i, \varphi) F_i\big(t, u_i(t - \tau_i, \varphi)\big) - u_i(t, \varphi) G_i(t, u_1, u_2)\\ &\geqslant -u_i(t, \varphi) G_i(t, u_1, u_2).\end{aligned}$$

Then

$$u_i(t, \varphi) \geqslant u_i(\tau_i + \theta_i, \varphi) e^{-\int_{\tau_i+\theta_i}^{t} G_i(s, u_1, u_2)\, ds} > 0 \quad \text{for } t \geqslant \tau_i + \theta_i.$$

Therefore, $u_i(t, \varphi) > 0$ for $t \geqslant \tau_i + \theta_i$. Thus $u(t, \varphi) \gg 0$ for $t \geqslant \tau = \max(\tau_1, \tau_2)$.

Given an order interval $I = [0, \alpha_1] \times [0, \alpha_2]$, $\alpha_i \in X_i^+$, $i = 1, 2$. $S^n(I)$ is precompact because of the compactness of S^n for $nT \geqslant \tau$ (see, e.g., [8, Theorem 3.6.1]). Thus, for all $nT \geqslant \tau$, S^n is order compact with respect to $\leqslant_K$.

At any point $\varphi = (\varphi_1, \varphi_2) \in \operatorname{int}(X^+)$, the Jacobi matrix of system (1.5) is

$$D(f_1, f_2) = \begin{pmatrix} D_{11} & -\varphi_1(0) \dfrac{\partial}{\partial u_2} G_1\big(t, \varphi_1(0), \varphi_2(0)\big) \\ -\varphi_2(0) \dfrac{\partial}{\partial u_1} G_2\big(t, \varphi_1(0), \varphi_2(0)\big) & D_{22} \end{pmatrix},$$

where

$$D_{ii} = \frac{\partial}{\partial u_i}\big(u_i F_i(t, u_i)\big)\bigg|_{u_i = \varphi_i(-\tau_i)} - \frac{\partial}{\partial u_i}\big(u_i G_i(t, u_1, u_2)\big)\bigg|_{u_1 = \varphi_1(0),\, u_2 = \varphi_2(0)},$$

$i = 1, 2$. $D(f_1, f_2)$ is irreducible due to assumption (H1). By [19, Theorem 5.3.4], it then easily follows that $S^n(\varphi) \ll_K S^n(\psi)$ $\forall nT \geqslant 3\tau$ for any $\varphi, \psi \in \operatorname{int}(X^+)$ with $\varphi <_K \psi$.

Let φ, ψ be in X^+ satisfying $\varphi = (\varphi_1, \varphi_2) \gg 0$, $\psi = (\psi_1, 0) \in X_1^+ \times \{0\}$, and $\varphi_1 \leqslant \psi_1$. Then $\varphi <_K \psi$. We want to show that $S^n(\varphi) \ll_K S^n(\psi)$ for all large integers n. Let $u(t, \varphi) = (u_1(t, \varphi), u_2(t, \varphi))$, $u(t, \psi) = (u_1(t, \psi), 0)$. Then $u(t, \varphi) \leqslant_K u(t, \psi)$, i.e., $0 \leqslant u_1(t, \varphi) \leqslant u_1(t, \psi)$, $u_2(t, \varphi) \geqslant 0$. By the above claim, we have $u_i(t, \varphi) > 0$ $\forall t \geqslant \tau$. Thus we only need to prove that $u_1(t,\varphi) < u_1(t,\psi)$ $\forall t > 0$. Assume, by contradiction, that $u_1(t_0,\varphi) = u_1(t_0,\psi)$ for some $t_0 > 0$. Since

$$\frac{\partial}{\partial u_2} G_1(t,u_1,u_2) > 0 \quad \text{and} \quad \frac{\partial}{\partial u_1} u_1 F_1(t,u_1) > 0,$$

we have

$$\begin{aligned}
&\dot u_1(t_0,\varphi) - \dot u_1(t_0,\psi)\\
&\quad = u_1(t_0-\tau_1,\varphi)F_1\big(t_0,u_1(t_0-\tau_1,\varphi)\big) - u_1(t_0-\tau_1,\psi)F_1\big(t_0,u_1(t_0-\tau_1,\psi)\big)\\
&\qquad + u_1(t_0,\psi)G_1\big(t_0,u_1(t_0,\psi),0\big) - u_1(t_0,\varphi)G_1\big(t_0,u_1(t_0,\varphi),u_2(t_0,\varphi)\big) < 0,
\end{aligned}$$

which implies that $u_1(t,\varphi) - u_1(t,\psi) > 0$ for some $t < t_0$. The conclusion contradicts $u_1(t,\varphi) \leqslant u_1(t,\psi)$ for all $t \geqslant -\tau_1$. Thus, $u_1(t,\varphi) < u_1(t,\psi)$ $\forall t > 0$, and hence we have $u(t,\varphi) \ll_K u(t,\psi)$ for $t > 0$. In particular, $S^n(\varphi) \ll_K S^n(\psi)$ for all $nT \geqslant 2\tau$. Similarly, if φ and ψ belong to X^+ and satisfy $\varphi <_K \psi$, $\psi \in \mathrm{int}(X^+)$ and $\varphi \in \{0\} \times X_2^+$, we have $S^n(\varphi) \ll_K S^n(\psi)$ for all $nT \geqslant 2\tau$.

Let us fix an integer n_0 such that S^{n_0} satisfies:

(1) $S^{n_0}(\varphi) \gg 0$ for any $\varphi \in X^0$.
(2) If $\varphi, \psi \in X^+$ satisfy $\varphi <_K \psi$, and either φ or ψ belongs to $\mathrm{int}(X^+)$, then $S^{n_0}(\varphi) \ll_K S^{n_0}(\psi)$.

Also, S^{n_0} has the following properties:

(3) S^{n_0} is order compact and strictly monotone with respect to $\leqslant_K$.
(4) $S^{n_0}(E_1) = E_1$ and

$$\lim_{n\to\infty} S^{n_0 n}\big((\varphi_1,0)\big) = E_1 \quad \text{for any } \varphi_1 \in X_1^+ \setminus \{0\}.$$

The symmetric results hold for E_2.
(5) Since $r_{12} > 1$, it follows from Lemma 3.2 that E_1 is an isolated fixed point of S^{n_0}, and $W^s(E_1) \cap \mathrm{int}(X^+) = \emptyset$, where $W^s(E_1)$ is the stable set of E_1 for S^{n_0}. The same results hold for E_0 and E_2. Also, Theorem 2.1 implies that E_0 is a repelling fixed point of S^{n_0}.

By the compression theorem [28, Theorem 2.4.2], we have the following results for S^{n_0}.

(i) S^{n_0} has two positive fixed points ϕ^* and ϕ^{**} with $\phi^{**} \leqslant_K \phi^*$. Then, system (1.5) has two positive n_0T-periodic solutions $u(t,\phi^*)$ and $u(t,\phi^{**})$ with $u(t,\phi^{**}) \leqslant_K u(t,\phi^*)$.
(ii) For every $\psi = (\psi_1,\psi_2) \in X^+$, $\psi_2 \neq 0$ and $\phi^* \leqslant_K \psi <_K E_1$, $\lim_{n\to\infty} S^{n_0 n}(\psi) = \phi^*$. It then follows that

$$\lim_{t\to\infty} \big\| u(t,\psi) - u(t,\phi^*)\big\| = 0.$$

Symmetrically, for every $\psi = (\psi_1,\psi_2) \in X^+$ with $\psi_1 \neq 0$ and $E_2 <_K \psi \leqslant_K \phi^{**}$,

$$\lim_{n\to\infty} S^{n_0 n}(\psi) = \phi^{**},$$

and hence,

$$\lim_{t\to\infty}\bigl\|u(t,\psi)-u(t,\phi^{**})\bigr\|=0.$$

(iii) Since

$$\lim_{n\to\infty}\operatorname{dist}\bigl(S^{n_0 n}(\psi),\ [\phi^{**},\phi^{*}]_K\bigr)=0 \quad \text{for any point } \psi\in X^0,$$

therefore,

$$\lim_{t\to\infty}\operatorname{dist}\bigl(u(t,\psi),\bigl[u(t,\phi^{**}),u(t,\phi^{*})\bigr]_K\bigr)=0.$$

Step 2. It remains to prove that $u(t,\phi^*)$ and $u(t,\phi^{**})$ are T-periodic solutions. We only need to show that ϕ^* and ϕ^{**} are fixed points of S. In what follows, we prove that ϕ^{**} is a fixed point for S.

By Proposition 2.1, we have $P_1^{(2)}e_1=r_{21}e_1$, and $e_1\gg 0$. Let S^{**} be the Poincaré map of the linearized system (3.7)–(3.8). We claim that r_{21} is an eigenvalue of S^{**}. Indeed, for any $\varphi\in X_2^+$, suppose that $u(t,\sigma,\varphi)$ solves

$$\dot u_2(t)=b_2^{(2)}u_2(t-\tau_2)-a_{22}^{(2)}u_2(t) \tag{3.11}$$

with initial values $u_\sigma=\varphi$. Let $W(t,\sigma)\varphi=u_t(\sigma,\varphi)$, then $W(t,\sigma)$ is a continuous linear evolution operator. Let $u_1(t,e_1)$ be the solution of Eq. (3.7) satisfying $u_1(\theta,e_1)=e_1(\theta)$ $\forall\theta\in[-\tau_1,0]$. By the variation-of-constants formula, the solutions of Eq. (3.8) can be expressed by

$$u_t(\sigma,\varphi)=W(t,\sigma)\varphi+\int_\sigma^t W(t,s)X_0h(s)\,ds,\quad t\geqslant\sigma,$$

where $X_0(\theta)=0$ for $\theta\in[-\tau_2,0)$, $X_0(\theta)=1$ for $\theta=0$, and $h(s)=-a_{21}^{(2)}(s)u_1(s,e_1)<0$.

Consider the following equation:

$$\bigl(r_{21}-W(T,0)\bigr)e_2=-\int_0^T W(T,s)X_0h(s)\,ds,\quad e_2\in X_2^+. \tag{3.12}$$

Since $u^{(2)}(t)$ is a globally asymptotically stable T-periodic solution of Eq. (3.4), and its linearized equation at $u^{(2)}(t)$ coincides with Eq. (3.11), we have $r(W(T,0))\leqslant 1$. Since $W(T,s)X_0>0$,

$$-\int_0^T W(T,s)X_0h(s)\,ds>0.$$

By the Krein–Rutman theorem (see, e.g., [11, Theorem 7.3]), Eq. (3.12) has a unique solution e_2 and $e_2\gg 0$. Let $e=(e_1,-e_2)$, then $e\gg_K 0$. Let P_2 be the Poincaré map of Eq. (3.8). Then,

$$P_2(-e_2)=W(T,0)(-e_2)+\int_0^T W(T,s)X_0h(s)\,ds.$$

Thus,

$$S^{**}e = \left(P_1^{(2)}(e_1), P_2(-e_2)\right) = r_{21}(e_1, -e_2) = r_{21}e,$$

and hence r_{21} is an eigenvalue of S^{**} with eigenfunction $e \gg_K 0$.

For any $\varepsilon > 0$ (note that $DS(E_2) = S^{**}$) we have

$$S(E_2 + \varepsilon e) = S(E_2) + DS(E_2)(\varepsilon e) + o(\varepsilon) = E_2 + \varepsilon\left(r_{21}e + \frac{o(\varepsilon)}{\varepsilon}\right).$$

Since $r_{21} > 1$, $(r_{21} - 1)e \in \operatorname{int}(K)$, there exists $\varepsilon_0 > 0$ such that $(r_{21} - 1)e + o(\varepsilon)/\varepsilon \in \operatorname{int}(K)$ for any $\varepsilon \in (0, \varepsilon_0]$. Hence $S(E_2 + \varepsilon e) - (E_2 + \varepsilon e) \gg_K 0$; that is, $E_2 + \varepsilon e \ll_K S(E_2 + \varepsilon e)$. Since S is monotone with respect to $\leqslant_K$, we have an increasing sequence $E_2 + \varepsilon e \ll_K S^n(E_2 + \varepsilon e) \leqslant_K S^{n+1}(E_2 + \varepsilon e)$ for all $n \geqslant 1$. Since $E_2 <_K \phi^{**}$ and $\phi^{**} \gg 0$, we can choose an ε such that $E_2 + \varepsilon e \leqslant_K \phi^{**}$. Therefore,

$$\lim_{n\to\infty} S^{n_0 n}(E_2 + \varepsilon e) = \phi^{**},$$

and hence

$$\lim_{n\to\infty} S^{n}(E_2 + \varepsilon e) = \phi^{**}.$$

By the continuity of S, it follows that ϕ^{**} is a fixed point of S. In the same way, it is easy to show that ϕ^{*} is a fixed point of S.

In the case of $\tau_i = k_i T$, if

$$\int_0^T \left(b_i(t) - a_i(t)\right) dt > 0,$$

$$\int_0^T \left(b_j^{(i)}(t) - b_{jj}^{(i)}(t)\right) dt > 0, \quad 1 \leqslant i \neq j \leqslant 2,$$

Proposition 2.1 implies the last statement in the theorem. $\square$

Theorem 3.1 implies that two species coexist. The following result shows that one species drives the other one to extinction.

Theorem 3.2. *Let* (H1) *and* (H2) *hold. Assume that system* (1.5) *has no positive T-periodic solution. If* (H3) *holds and $r_{21} > 1$, or in the case where $\tau_i = k_i T$ for some integers k_i, if*

$$\int_0^T \left(b_i(t) - a_i(t)\right) dt > 0 \quad \forall i = 1, 2, \quad \textit{and} \quad \int_0^T \left(b_1^{(2)}(t) - a_{11}^{(2)}(t)\right) dt > 0,$$

then for any $\psi \in X^0$, the solution $u(t, \psi)$ of system (1.5) *satisfies*

$$\lim_{t\to\infty} \left\| u(t, \psi) - \left(u^{(1)}(t), 0\right) \right\| = 0.$$

A symmetric result holds for $(0, u^{(2)}(t))$.

Proof. In the case of $r_{21} > 1$, by Lemma 3.2, for any $\psi \in X^0$, the omega limit set $\omega(\psi)$ of $S^n(\psi)$ cannot be E_2 since $S^n(\psi) \gg 0$ for all $nT \geqslant 2\tau$ (see the claim in the proof of Theorem 3.1). Moreover, just as in the proof of Theorem 3.1, we can consider S^{n_0} such that S^{n_0} satisfies the assumptions of Theorem A in [12]. Note that system (1.5) has no positive T-periodic solutions, and hence S has no positive fixed points, we have $S^{n \cdot n_0}(\psi) \to E_1 (n \to \infty)$. Therefore,

$$\lim_{t\to\infty} \| u(t,\psi) - u(t,E_1) \| = \lim_{t\to\infty} \| u(t,\psi) - \big(u^{(1)}(t), 0\big) \| = 0.$$

A symmetric result holds for $(0, u^2(t))$. □

In practice, it is not easy to verify the nonexistence of positive T-periodic solutions. To establish some sufficient conditions for the conclusion of Theorem 3.2, we assume that

(H4) $f_1(t,\cdot,u_2,\cdot)$ and $f_2(t,u_1,\cdot,\cdot)$ are strictly sublinear on $\mathbb{R}^2_+$, where $u_1, u_2 \geqslant 0$, and $f_1(t,L,0,L) \leqslant 0$, $f_2(t,0,L,L) \leqslant 0$ for some $L > 0$.

Then assumption (H1) implies $f_1(t,L,u_2,L) \leqslant 0$, $f_2(t,u_1,L,L) \leqslant 0$ for all $u_1, u_2 \geqslant 0$. By Theorem 2.1, if $r_{21} > 1$, then equation

$$\dot{u}_1(t) = f_1\big(t, u_1(t), u^{(2)}(t), u_1(t-\tau_1)\big)$$

admits a unique positive T-periodic solution $u_1^{(2)}(t)$, which is globally asymptotically stable with respect to $X_1^+ \setminus \{0\}$, where $u^{(2)}(t)$ is the positive T-periodic solution of Eq. (3.4). Let $r_{1,2}^{(2)}$ be the spectral radius, defined by Theorem 2.1, associated with

$$\dot{u}_2(t) = f_2\big(t, u_1^{(2)}(t), u_2(t), u_2(t-\tau_2)\big).$$

Then we have the following result.

Corollary 3.1. *Let* (H1), (H3) *and* (H4) *hold. Then if* $r_{21} > 1$ *and* $r_{1,2}^{(2)} < 1$, *the conclusion of Theorem* 3.2 *holds.*

Proof. We use the same notation as in Theorem 3.2. Assumption (H3) implies $u^{(2)}(t)$ is globally asymptotically stable with respect to $X_2^+ \setminus \{0\}$ for Eq. (3.4). For any $\psi \in X^0$, let $u(t,\psi) = (u_1(t), u_2(t))$. Since assumption (H1) implies

$$\dot{u}_2(t) = f_2\big(t, u_1(t), u_2(t), u_2(t-\tau_2)\big) \leqslant f_2\big(t, 0, u_2(t), u_2(t-\tau_2)\big),$$

for any small $\varepsilon > 0$, we have $u_2(t) < u^{(2)}(t) + \varepsilon$ for all $t > t(\varepsilon)$. Therefore,

$$\dot{u}_1(t) = f_1\big(t, u_1(t), u_2(t), u_1(t-\tau_1)\big) > f_1\big(t, u_1(t), u^{(2)}(t) + \varepsilon, u_1(t-\tau_1)\big) \tag{3.13}$$

for $t > t(\varepsilon)$. Let r_{21}^{ε} be the spectral radius defined by Theorem 2.1, as applied to

$$\dot{u}(t) = f_1\big(t, u(t), u^{(2)}(t) + \varepsilon, u(t-\tau_1)\big). \tag{3.14}$$

Then $\lim_{\varepsilon\to 0} r_{21}^{\varepsilon} = r_{21} > 1$, and hence $r_{21}^{\varepsilon} > 1$ for all sufficiently small ε. Therefore, by assumption (H4) and Theorem 2.1, there exists a unique positive T-periodic solution $u_1^{\varepsilon}(t)$

for Eq. (3.14), and $u_1^\varepsilon(t)$ is globally asymptotically stable with respect to $X_1^+ \setminus \{0\}$. By inequality (3.13), it follows that for any $\varepsilon' > 0$, we have $u_1(t) > u_1^\varepsilon(t) - \varepsilon'$ for $t > t(\varepsilon, \varepsilon')$. Therefore, assumption (H1) implies

$$\dot{u}_2(t) = f_2\big(t, u_1(t), u_2(t), u_2(t-\tau_2)\big) < f_2\big(t, u_1^\varepsilon(t) - \varepsilon', u_2(t), u_2(t-\tau_2)\big) \tag{3.15}$$

for $t > t(\varepsilon, \varepsilon')$. Let $r^{\varepsilon'}$ be the spectral radius defined by Theorem 2.1, as applied to

$$\dot{u}(t) = f_2\big(t, u_1^\varepsilon(t) - \varepsilon', u(t), u(t-\tau_2)\big). \tag{3.16}$$

Note that $\lim_{\varepsilon\to 0} u_1^\varepsilon(t) = u_1^{(2)}(t)$ uniformly for $t \in [0, T]$ (see, e.g., [28, Theorem 1.4.1] or [21, Theorem 2.1]). We have $\lim_{\varepsilon,\varepsilon'\to 0} r^{\varepsilon'} = r_{1,2}^{(2)} < 1$, and hence $r^{\varepsilon'} < 1$ for all sufficiently small ε and ε'. Therefore, by Theorem 2.1, zero solution is globally asymptotically stable for Eq. (3.16). Thus inequality (3.15) implies $\lim_{t\to\infty} u_2(t) = 0$. That is, system (1.5) has no positive T-periodic solutions. Therefore, Theorem 3.2 completes the proof. □

Remark 3.1. Theorem 3.1, as applied to system (1.3) with $n = 2$, implies that system (1.3) is permanent and has at least one positive T-periodic solution. In particular, if there is only one positive T-periodic solution, then it is globally attractive. Therefore, the conclusions of Theorem 3.1 are stronger than [17, Theorem 2.2] for system (1.3) with $n = 2$. Furthermore, since assumptions (H1)–(H3) are automatically satisfied for system (1.3), Theorem 3.1 holds if $r_{12} > 1$, $r_{21} > 1$, or if

$$\int_0^T \big(b_2^{(1)}(t) - a_{22}^{(1)}(t)\big)\,dt > 0 \quad \text{and} \quad \int_0^T \big(b_1^{(2)}(t) - a_{11}^{(2)}(t)\big)\,dt > 0$$

in the case of $\tau_i = k_i T$, $i = 1, 2$.

Remark 3.2. For system (1.3) with $n = 2$, the conditions of [17, Theorem 2.2] are sufficient for $r_{12} > 1$ and $r_{21} > 1$ (see Lemma 3.3). Thus, Theorem 3.1 is a natural generalization of [17, Theorem 2.2].

Lemma 3.3. *If inequalities* (1.4) *hold, then* $r_{12} > 1$ *and* $r_{21} > 1$.

Proof. For system (1.3) with $n = 2$, the corresponding Eqs. (3.3) and (3.6) reduce to

$$\dot{x}_1(t) = B_1(t)x_1(t-\tau_1) - a_{11}x_1^2(t), \tag{3.17}$$

$$\dot{x}_2(t) = B_2(t)x_2(t-\tau_2) - a_{21}(t)u^{(1)}(t)x_2(t), \tag{3.18}$$

respectively, where $u^{(1)}(t)$ is the positive T-periodic solution for Eq. (3.17). Note that $u^{(1)}(t)$ is globally asymptotically stable with respect to $X_1^+ \setminus \{0\}$, and that r_{12} is the spectral radius of the Poincaré map $P_2^{(1)}$ associated with Eq. (3.18). Choosing t^* such that $u^{(1)}(t^*) = \max_{t\in[0,T]} u^{(1)}(t)$, we then have

$$0 = \dot{u}^{(1)}(t^*) = B_1(t^*)u^{(1)}(t^* - \tau_1) - a_{11}(t^*)\big(u^{(1)}(t^*)\big)^2.$$

Therefore,

$$a_{11}(t^*)\big(u^{(1)}(t^*)\big)^2 = B_1(t^*)u^{(1)}(t^* - \tau_1) \leqslant B_1(t^*)u^{(1)}(t^*),$$

and hence $u^{(1)}(t^*) \leqslant B_1^m/a_{11}^l$, where by the upper indexes we mean the same as these in inequalities (1.4).

By inequalities (1.4), it is easy to see that for any $\varphi \in X_2^+$ with $\varphi \gg 0$, the solution $x(t,\varphi)$ of the equation

$$\dot{x}(t) = B_2^l x(t-\tau_2) - a_{21}^m \frac{B_1^m}{a_{11}^l} x(t)$$

satisfies $\lim_{t\to\infty} x(t,\varphi) = \infty$. By the proof of Proposition 2.1, it follows that Eq. (3.18) has a positive solution $u^0(t) = v_0(t)e^{\lambda_0 t}$ with $\lambda_0 = (1/T)\ln r_{12}$ and $v_0(t)$ being continuous and T-periodic.

Let $\varphi_0(s) = u^0(s)$, $s \in [-\tau_2, 0]$, then $\varphi_0 \gg 0$. Note that

$$\dot{x}_2(t) = B_2(t)x_2(t-\tau_2) - a_{21}(t)u^{(1)}(t)x_2(t) \geqslant B_2^l x_2(t-\tau_2) - a_{21}^m \frac{B_1^m}{a_{11}^l} x_2(t).$$

By the comparison theorem, we have $u^0(t) \geqslant x(t,\varphi_0)$, and hence $\lim_{t\to\infty} u^0(t) = \infty$. This implies that $\lambda_0 > 0$ and hence $r_{12} > 1$. By a similar argument, we have $r_{21} > 1$. □

Remark 3.3. Theorem 3.2 and Corollary 3.1 imply that one species persists at a positive periodic solution while the other one dies out. The conclusion of Corollary 3.1, as applied to system (1.3) with $n = 2$, is the same as [17, Corollaries 2.1 and 2.2]. However, by the comparison method in the proofs of Lemma 3.3 and Corollary 3.1, one can easily conclude that the conditions in [17, Corollaries 2.1 and 2.2] are sufficient for the conditions in Corollary 3.1.

4. Multi-species competition

As we have seen in Section 3, the monotonicity of the Poincaré map associated with the periodic system (1.5) with $m = 2$ plays an important role in obtaining the global dynamics. However, for system (1.5) with $m \geqslant 3$, we are not able to appeal to the powerful theory of monotone dynamical systems. In this section, we use the elementary comparison method to establish a set of conditions for uniform persistence in system (1.5) with $m \geqslant 3$. In virtue of the persistence theory, we further obtain natural invasibility conditions for uniform persistence and the existence of positive periodic solutions in 3-species competitive periodic system (1.5).

We first consider m-species competitive system (1.5). Assume that for $u_i \geqslant 0$, $1 \leqslant i \neq j \leqslant m$, we have

(S1) $F_i(t,u_i) > 0$, $(\partial/\partial u_i)(u_i F_i(t,u_i)) > 0$, $(\partial/\partial u_j)G_i(t,u_1,\dots,u_m) > 0$;

(S2) $f_i(t,u_1,\dots,u_{i-1},\cdot,u_{i+1},\dots,u_m,\cdot)$ is strictly sublinear on $\mathbb{R}_+^2$; and for some $L > 0$, $f_i(t,0,\dots,0,L,0,\dots,0,L) \leqslant 0$, where L is at the positions of the ith and $(m+1)$th arguments of f_i except for t.

Then $f_i(t,u_1,\dots,u_{i-1},L,u_{i+1},\dots,u_m,L) \leqslant 0$, for all $u_i \geqslant 0$, $l \geqslant L$, $i = 1,2,\dots,m$. As analyzed before, it easily follows that solutions of system (1.5) are uniformly bounded.

Let $\bar{r}_i$ be the spectral radius defined by Theorem 2.1, as applied to the scalar periodic equation

$$\dot{u}_i(t) = f_i\big(t, 0, \ldots, 0, u_i(t), 0, \ldots, 0, u_i(t-\tau_i)\big). \tag{4.1}$$

Assume that

(S3) $\bar{r}_i > 1$, $i = 1, 2, \ldots, m$.

Then for each i, by Theorem 2.1, (4.1) admits a unique positive T-periodic solution $\bar{u}_i(t, \bar{\phi}_i)$, which attracts every solution of (4.1) except for zero. Let $\underline{r}_i$ be the spectral radius defined by Theorem 2.1, as applied to the scalar periodic equation

$$\begin{aligned} \dot{u}(t) = f_i\big(&t, \bar{u}_1(t, \bar{\phi}_1), \ldots, \bar{u}_{i-1}(t, \bar{\phi}_{i-1}), u_i(t), \\ &\bar{u}_{i+1}(t, \bar{\phi}_{i+1}), \ldots, \bar{u}_m(t, \bar{\phi}_m), u(t-\tau_i)\big). \end{aligned} \tag{4.2}$$

If we assume that $\underline{r}_i > 1$, then (4.2) admits a unique positive T-periodic solution $\underline{u}_i(t, \underline{\phi}_i)$, which attracts all solutions of Eq. (4.2) except for zero.

Let

$$Z_m^+ = C\left(\prod_{i=1}^m [-\tau_i, 0], \mathbb{R}_+^m\right),$$
$$Z_m^0 = \big\{\psi = (\psi_i)_{i=1}^m \in Z_m^+ \colon\ \psi_i \neq 0\ \forall i,\ 1 \leqslant i \leqslant m\big\}.$$

For any $\phi \in Z_m^+$, let us denote by $u(t, \psi) = (u_i(t))_{i=1}^m$ the solution of system (1.5) with $u_0(\psi) = \psi$. The following theorem implies that system (1.5) is uniformly persistent.

Theorem 4.1. *Let assumptions* (S1)–(S3) *hold. Suppose that* $\underline{r}_i > 1$, $i = 1, 2, \ldots, m$. *Then system* (1.5) *admits a positive T-periodic solution, and for any $\psi \in Z_m^0$, the solution $u(t, \psi)$ of system* (1.5) *satisfies*

$$\lim_{t\to\infty} \operatorname{dist}\big(u(t, \psi), \big[\underline{u}(t), \bar{u}(t)\big]\big) = 0,$$

where

$$\big[\underline{u}(t), \bar{u}(t)\big] = \big\{u = (u_i)_{i=1}^m \in \mathbb{R}_+^m \colon\ \underline{u}_i(t, \underline{\phi}_i) \leqslant u_i \leqslant \bar{u}_i(t, \bar{\phi}_i)\ \forall 1 \leqslant i \leqslant m\big\}.$$

Proof. By Theorem 2.1 and the standard two-side comparison method similar to that in the proof of Corollary 3.1, for any $\psi \in Z_m^0$ and any small $\varepsilon, \varepsilon' > 0$, we have $\underline{u}_i^\varepsilon(t) - \varepsilon' < u_i(t, \psi) < \bar{u}_i(t, \bar{\phi}_i) + \varepsilon$ for $t > t(\varepsilon, \varepsilon')$, where $\underline{u}_i^\varepsilon(t)$ is positive and T-periodic and satisfies that

$$\lim_{\varepsilon\to 0} \underline{u}_i^\varepsilon(t) = \underline{u}_i(t, \underline{\phi}_i) \quad \text{uniformly for } t \in [0, T].$$

Let $\varepsilon, \varepsilon' \to 0$, we have

$$\lim_{t\to\infty} \operatorname{dist}\big(u(t, \psi), \big[\underline{u}(t), \bar{u}(t)\big]\big) = 0.$$

Let S be the Poincaré map of system (1.5). Then S is bounded, point dissipative, α-condensing and uniformly persistent with respect to $(Z_m^0, \partial Z_m^0)$, where $\partial Z_m^0 = Z_m^+ \setminus Z_m^0$.

Furthermore, S^n is compact for $nT \geqslant 2\tau = 2\max(\tau_1, \tau_2, \ldots, \tau_m)$. By [28, Theorem 1.3.6 and Remark 3.1.1], S has a coexistence state $\phi_0 \in Z_m^0$. Thus system (1.5) admits a positive T-periodic solution $u(t, \phi_0)$. $\square$

As mentioned in [23], for the periodic system (1.5) in the case of $m = 2$, $(\bar{u}_1(t, \bar{\phi}_1), 0)$ and $(0, \bar{u}_2(t, \bar{\phi}_2))$ (i.e., $(u^{(1)}(t), 0)$ and $(0, u^{(2)}(t))$ defined in Section 3) are the semitrivial periodic solutions. Then $\underline{r}_1 > 1$ and $\underline{r}_2 > 1$ (i.e., $r_{12} > 1$, $r_{21} > 1$ in Theorem 3.1) are natural invasibility conditions for uniform persistence. However, for the m-species competition periodic system ($m \geqslant 3$), $(\bar{u}_1(t, \bar{\phi}_1), \ldots, \bar{u}_{i-1}(t, \bar{\phi}_{i-1}), 0, \bar{u}_i(t, \bar{\phi}_i), \ldots, \bar{u}_m(t, \bar{\phi}_m))$ ($1 \leqslant i \leqslant m$) are not solutions of system (1.5), and hence, due to the overestimation of the effect of competition, conditions $\underline{r}_i > 1$ in Theorem 4.1 are very strong conditions. In the rest of this section, we use the ideas in [23] to obtain some natural invasibility conditions for uniform persistence and existence of a positive coexistence state in the three-species competition.

Consider the T-periodic model for the three-species competition

$$\begin{aligned} \dot{u}_i(t) &= u_i(t-\tau_i) F_i\big(t, u_i(t-\tau_i)\big) - u_i(t) G_i\big(t, u_1(t), u_2(t), u_3(t)\big) \\ &= f_i\big(t, u_1(t), u_2(t), u_3(t), u_i(t-\tau_i)\big), \quad 1 \leqslant i \leqslant 3, \end{aligned} \tag{4.3}$$

which satisfies conditions (S1)–(S3) in the case of $m = 3$. For each i, there is a corresponding 2-species competition system

$$\dot{u}_j(t) = f_j\big(t, u_1(t), u_2(t), u_3(t), u_j(t-\tau_j)\big), \quad u_i(t) \equiv 0,\ j \neq i,\ 1 \leqslant j \leqslant 3. \tag{R_i}$$

Suppose that each system (R_i) satisfies the conditions either in Theorem 3.1 or in Theorem 3.2. We consider the following three cases:

(Q1) Each (R_i) satisfies Theorem 3.1 and admits only one positive T-periodic solution $\hat{u}^{(i)}(t)$.
(Q2) both (R_2) and (R_3) satisfy Theorem 3.1, and each of them admits only one positive T-periodic solution. (R_1) satisfies Theorem 3.2.
(Q3) (R_3) satisfies Theorem 3.1 and admits only one positive T-periodic solution. (R_1) and (R_2) satisfy Theorem 3.2.

Let

$$Z_3^+ = C\left(\prod_{i=1}^{3}[-\tau_i, 0], \mathbb{R}_+^3\right), \qquad Z_3^0 = \big\{(\phi_i)_{i=1}^3 \in Z_3^+ \colon \phi_i \neq 0\ \forall 1 \leqslant i \leqslant 3\big\}.$$

For any $\phi \in Z_3^+$, denote the solution of system (4.3) by $u(t, \phi) = (u_i(t, \phi))_{i=1}^3$, and the solution semiflow by $u_t(\phi)$. We then have the following results.

Theorem 4.2. *Let* (Q1) *hold. Denote by* $\underline{r}^{(1)}$ *the spectral radius defined by Theorem* 2.1, *as applied to* $\dot{u}(t) = f_1(t, u(t), \hat{u}^{(1)}(t), u(t-\tau_1))$. *In the same way, we can define* $\underline{r}^{(i)}$, $i = 2, 3$. *Suppose that* $\underline{r}^{(i)} > 1$, $i = 1, 2, 3$. *Then system* (4.3) *admits a positive* T*-periodic solution and is permanent in the sense that there exist* $\alpha > 0$ *and* $\beta > 0$ *such that for any* $\phi \in Z_3^0$, $\beta \leqslant \liminf_{t\to\infty} u_i(t, \phi) \leqslant \limsup_{t\to\infty} u_i(t, \phi) \leqslant \alpha$.

Proof. For any $\phi \in Z_3^0$, by the argument similar to the claim in the proof of Theorem 3.1, $u_i(t,\phi) > 0$ for all $t \geqslant \tau = \max(\tau_1, \tau_2, \tau_3)$. For any $\phi \in Z_3^+$, define $\mathcal{T}(t)(\phi) := u_t(\phi)$, $S(\phi) := u_T(\phi)$. Then $\mathcal{T}(t)\phi, S^n(\phi) \in \operatorname{int}(Z_3^+)$ for $\phi \in Z_3^0$ and $t, nT \geqslant 2\tau$. By the same argument as in the proof of Corollary 3.1, we have $u_i(t,\phi) < \bar{u}_i(t, \bar{\phi}_i) + \varepsilon\ \forall t > t(\varepsilon)$. Thus, it is easy to find a number α such that

$$\limsup_{t\to\infty} u_i(t,\phi) \leqslant \alpha \quad \text{for all } i \text{ and } \phi \in Z_3^0.$$

In particular, S is point dissipative and a bounded map (by the same argument as in Lemma 3.1).

Note that S is α-condensing and orbits of bounded sets are bounded. By [9, Theorem 2.4.7], S admits a connected global attractor $A \subset Z_3^+$. Let $M_1 = (0,0,0)$, $M_2 = (\bar{\phi}_1, 0, 0)$, $M_3 = (0, \bar{\phi}_2, 0)$, $M_4 = (0, 0, \bar{\phi}_3)$, $M_5 = (0, \hat{\phi}_2^{(1)}, \hat{\phi}_3^{(1)})$, $M_6 = (\hat{\phi}_1^{(2)}, 0, \hat{\phi}_3^{(2)})$, $M_7 = (\hat{\phi}_1^{(3)}, \hat{\phi}_2^{(3)}, 0)$, where $(\hat{\phi}_2^{(1)}, \hat{\phi}_3^{(1)})$, $(\hat{\phi}_1^{(2)}, \hat{\phi}_3^{(2)})$, and $(\hat{\phi}_1^{(3)}, \hat{\phi}_2^{(3)})$ are initial functions of $\hat{u}^{(1)}(t), \hat{u}^{(2)}(t)$ and $\hat{u}^{(3)}(t)$, respectively. Clearly, all M_i are fixed points of S. For any $\phi \in \partial Z_3^0 = Z_3^+ \setminus Z_3^0$, let $\omega(\phi)$ be the ω-limit set of ϕ with respect to the discrete semiflow $\{S^n\}_{n=0}^{\infty}$. By assumption (Q1) and Theorem 3.1,

$$\bigcup_{\phi \in \partial Z_3^0} \omega(\phi) = \{M_1, M_2, M_3, M_4, M_5, M_6, M_7\},$$

and no subset of the M_i's forms a cycle for S in ∂Z_3^0. By assumptions (Q1) and (S1), simply following the proof of Lemma 3.2, we can obtain that M_i are isolated invariant sets in Z_3^+ for S, and $W^s(M_i) \cap \operatorname{int}(Z_3^+) = \emptyset$, where $W^s(M_i)$ is the stable set of M_i for S. Then $W^s(M_i) \cap Z_3^0 = \emptyset$. By [27, Theorem 2.2], it follows that S is uniformly persistent with respect to $(Z_3^0, \partial Z_3^0)$. Note that S^n is compact for $nT \geqslant 2\tau$, by [28, Theorem 1.3.6 and Remark 3.1.1], there exists a global attractor $A_0 \subset Z_3^0$ for $S: Z_3^0 \to Z_3^0$, and S admits a coexistence state $\phi_0 \in A_0$. Since $\phi_0 \in A_0 = S^n(A_0) \subset \operatorname{int}(Z_3^+)$ for $nT \geqslant 2\tau$, system (4.3) admits a positive T-periodic solution $u(t, \phi_0)$.

Let

$$A_0^* = \bigcup_{0 \leqslant t \leqslant n_0 T} \mathcal{T}(t) A_0, \quad \text{where } n_0 T \geqslant 2\tau.$$

Then by the argument given in the claim in the proof of Theorem 3.1, $A_0^* \in \operatorname{int}(Z_3^+)$, and by [27, Theorem 2.1], it follows that A_0^* is a compact set and attracts strongly bounded sets in Z_3^0. Since $\mathcal{T}(t)\phi \in \operatorname{int}(Z_3^+)$ for $t \geqslant 2\tau$ and $\phi \in Z_3^0$, A_0^* attracts every point in Z_3^0 under $\mathcal{T}(t)$. For every $\phi \in A_0^*$, there exists a number $\beta_\phi > 0$ such that $\phi \gg \beta_\phi I_d$, where $I_d = (1,1,1)$. By the compactness of A_0^*, it follows that there exists $\beta = \beta(V)$ such that $\phi \gg \beta I_d\ \forall \phi \in V$, where V is a neighborhood of A_0^* in $\operatorname{int}(Z_3^+)$. Thus for any $\phi \in Z_3^0$, $\mathcal{T}(t)\phi \gg \beta I_d$ for sufficiently large t, which implies that $\liminf_{t\to\infty} u_i(t,\phi) \geqslant \beta$. $\square$

Theorem 4.3. *Let* (Q_2) *hold, and* r_{32} *be spectral radius defined by Theorem* 3.2, *as applied to* (R_1). *Suppose that* $r_{32} > 1$, $\underline{r}^{(i)} > 1$, $i = 2, 3$. *Then the conclusions of Theorem* 4.2 *hold.*

Proof. We use the same notation as in the proof of Theorem 4.2. By Theorem 3.2, it follows that

$$\lim_{n\to\infty} S^n(\phi) = (0, \bar{\phi}_2, 0) = M_3$$

for any $\phi = (\phi_i)_{i=1}^3 \in \partial Z_3^0$ with $\phi_1 = 0$ and $\phi_2 \neq 0$. By assumption (Q2), Theorems 3.1 and 3.2,

$$\bigcup_{\phi\in\partial Z_3^0} = \{M_1, M_2, M_3, M_4, M_6, M_7\},$$

and no subset of the M_i's forms a cycle for S^{n_0} in ∂Z_3^0. As in the proof of Theorem 4.2, we have S is uniformly persistent with respect to $(Z_3^0, \partial Z_3^0)$. Now, the same argument as in Theorem 4.2 completes the proof. □

Theorem 4.4. *Let* (Q3) *hold and* r_{31} *be spectral radius defined by Theorem* 3.2*, as applied to* (R_2)*. Suppose that* $r_{31} > 1$*,* $r_{32} > 1$ *and* $\underline{r}^{(3)} > 1$*. Then the conclusions of Theorem* 4.2 *hold.*

Proof. We use the same notation as in the proof of Theorem 4.2. As in the proof of Theorem 4.3, assumption (Q3) implies that for any $\phi = (\phi_i)_{i=1}^3 \in \partial Z_3^0$ with $\phi_1 = 0$ and $\phi_2 \neq 0$

$$\lim_{n\to\infty} S^n(\phi) = (0, \bar{\phi}_2, 0) = M_3,$$

and for any $\phi = (\phi_i)_{i=1}^3 \in \partial Z_3^0$ with $\phi_2 = 0$ and $\phi_1 \neq 0$

$$\lim_{n\to\infty} S^n(\phi) = (\bar{\phi}_1, 0, 0) = M_2.$$

Clearly,

$$\bigcup_{\phi\in\partial Z_3^0} = \{M_1, M_2, M_3, M_4, M_7\}.$$

Then as in the proof of Theorem 4.2, S is uniformly persistent with respect to $(Z_3^0, \partial Z_3^0)$. Now, the same argument as in Theorem 4.2 completes the proof. □

References

[1] W.G. Aiello, H.I. Freedman, A time-delay model of single species growth with stage structure, Math. Biosci. 101 (1990) 139–153.

[2] W.G. Aiello, H.I. Freedman, J. Wu, Analysis of a species representing stage-structured populations growth with stage-dependent time delay, SIAM J. Appl. Math. 52 (1992) 855–869.

[3] F.J. Ayala, M.E. Gilpin, J.G. Ehrenfeld, Competition between species: theoretical models and experimental tests, Theoret. Popul. Biol. 4 (1973) 331–356.

[4] H.J. Barclay, P. van den Driessche, A model for a species with two life history stages and added mortality, Ecol. Model. 11 (1980) 157–166.

[5] J.M. Cushing, An Introduction to Structured Population Dynamics, Regional Conf. Ser. Appl. Math., vol. 71, SIAM, Philadelphia, PA, 1998.

[6] M. Fan, K. Wang, Global periodic solutions of a generalized n-species Gilpin–Ayala competition model, Comput. Math. Appl. 40 (2000) 1141–1151.
[7] H.I. Freedman, J. Wu, Persistence and global asymptotic stability of single species dispersal models with stage structure, Quart. Appl. Math. 5 (49) (1991) 351–371.
[8] J.K. Hale, Theory to Functional Differential Equations, Appl. Math. Sci., vol. 3, Springer-Verlag, New York, 1977.
[9] J.K. Hale, Asymptotic Behavior of Dissipative Systems, Math. Surveys Monogr., vol. 25, Amer. Math. Soc., Providence, RI, 1988.
[10] A. Hastings, Age-dependent predation is not a simple process, I. Continuous time models, Theoret. Popul. Biol. 23 (1983) 347–362.
[11] P. Hess, Periodic-Parabolic Boundary Value Problems and Positivity, Pitman Res. Notes Math. Ser., vol. 247, Longman, Harlow, 1991.
[12] S.B. Hsu, H.L. Smith, P. Waltman, Competitive exclusion and coexistence for competitive systems on ordered Banach spaces, Trans. Amer. Math. Soc. 348 (1996) 4083–4094.
[13] M. Kshatriya, C. Cosner, A continuum formulation of the ideal free distribution and its implications for population dynamics, Theor. Popul. Biol. 61 (2002) 277–284.
[14] H.D. Landahl, B.D. Hanson, A three-stage population model with cannibalism, Bull. Math. Biol. 37 (1975) 11–17.
[15] S. Liu, L. Chen, Extinction and permanence in competitive stage-structured system with time-delay, Nonlinear Anal. 51 (2002) 1347–1361.
[16] S. Liu, L. Chen, et al., Asymptotic behaviors of competitive Lotka–Volterra system with stage structure, J. Math. Anal. Appl. 271 (2002) 124–138.
[17] S. Liu, L. Chen, Z. Liu, Extinction and permanence in nonautonomous competitive system with stage structure, J. Math. Anal. Appl. 274 (2002) 667–684.
[18] J.A.J. Metz, O. Diekmann, The Dynamics of Physiologically Structured Populations, Lecture Notes in Biomath., vol. 68, Springer-Verlag, New York, 1986.
[19] H.L. Smith, Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems, Math. Surveys Monogr., vol. 41, Amer. Math. Soc., Providence, RI, 1995.
[20] H.L. Smith, H.R. Thieme, Strongly order preserving semiflows generated by functional differential equations, J. Differential Equations 93 (1991) 332–363.
[21] H.L. Smith, P. Waltman, Perturbation of a globally stable steady state, Proc. Amer. Math. Soc. 127 (1999) 447–453.
[22] J. So, J. Wu, X. Zou, Structured population on two patches: modeling dispersion and delay, J. Math. Biol. 43 (2001) 37–51.
[23] G.S.K. Wolkowicz, X.-Q. Zhao, N-species competition in a periodic chemostat, Differential Integral Equations 11 (1998) 465–491.
[24] D. Xu, Global dynamics and Hopf bifurcation of a structured population model, Nonlinear Anal. Real World Appl. 6 (2005) 461–476.
[25] X.-Q. Zhao, Global attractivity in monotone and subhomogeneous almost periodic systems, J. Differential Equations 187 (2003) 494–509.
[26] X.-Q. Zhao, Global attractivity and stability in some monotone discrete dynamical systems, Bull. Austral. Soc. 53 (1996) 305–324.
[27] X.-Q. Zhao, Uniform persistence and coexistence states in infinite-dimensional periodic semiflows applications, Canad. Appl. Math. Quart. 3 (1995) 473–495.
[28] X.-Q. Zhao, Dynamical Systems in Population Biology, CMS Books in Math. Ser., vol. 16, Springer-Verlag, New York, 2003.
[29] X.-Q. Zhao, Z.-J. Jing, Global asymptotic behavior in some cooperative systems of functional differential equations, Canad. Appl. Math. Quart. 4 (1996) 421–444.