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Erratum to "Bistable Waves in an Epidemic Model" [*J. Dynam. Diff. Eq.* 16, 679–707 (2004)][*,1]

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As originally printed, this article contained some incorrect reference numbers. We regret this error and apologize for any inconvenience caused. The article is reprinted here with the corrected references.

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The existence, uniqueness up to translation and global exponential stability with phase shift of bistable travelling waves are established for a quasi-monotone reaction–diffusion system modelling man–environment–man epidemics. The methods involve phase space investigation, monotone semiflows approach and spectrum analysis.

KEY WORDS: Epidemic model; travelling waves; monotone semiflows; spectrum analysis; global exponential stability.

1. INTRODUCTION

The geographic spread of infectious diseases is an important subject in mathematical epidemiology. To model the cholera epidemic which spread in the European Mediterranean regions in 1973, Capasso and Paveri-Fontana [4] proposed a system of two ordinary differential equations. As a basic feature, this model involves a positive feedback interaction between the infective human population and the concentration of bacteria. The human population, once infected, has a contribution to the growth rate of bacteria, which is then returned to the environment to increase the infection rate of humans. This kind of mechanisum seems to be appopriate to interpret other fecally–orally transmitted epidemics such as typhoid fever, infections hepatitis, polyometitis etc., with suitable modifications. Under the assumption that the bacteria disperse randomly while the small mobility of the infective human population is neglected, Capasso and Maddalena [5] further obtained a reaction–diffusion system

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$$\frac{\partial U_1}{\partial t}(x,t) = d\frac{\partial^2 U_1}{\partial x^2}(x,t) - a_{11}U_1(x,t) + a_{12}U_2(x,t),$$
$$\frac{\partial U_2}{\partial t}(x,t) = -a_{22}U_2(x,t) + g(U_1(x,t)). \tag{1.1}$$

Here d, a_{11}, a_{12} and a_{22} are the positive constants, $U_1(x,t)$ and $U_2(x,t)$, respectively, denote the spatial densities of infectious agent and the infective human population at a point x in the habitat at time $t \geqslant 0$, $1/a_{11}$ is the mean lifetime of the agent in the environment, $1/a_{22}$ is the infectious period of the human infections, a_{12} is the multiplicative factor of the mean infectious agent due to the human populations, and $g(x)$ is the infection rate of human under the assumption that total susceptible human population is constant during the evolution of the epidemic.

System (1.1) and its corresponding reaction system have received investigations. For example, the case in which there is at most one nontrivial endemic equilibrium was studied in [3–5,7], and it is known that above some parameter threshold a unique nontrivial state exists and all epidemic outbreaks tend to it (i.e., monostable case), below the parameter threshold, all epidemics tend to extinction. In [6], the bistable case (where the corresponding reaction system of (1.1) admits exactly toe nontrivial steady states) was obtained by assuming that the infection rate g is sigma-shaped. A saddle point structure was obtained in [6] for (1.1) with Neumann boundary conditions and its reaction systems, and a complete analysis of the steady states of (1.1) subject to Dirichlet boundary conditions and numerical simulations were made in [8]. It was shown in [23] that system (1.1) subject to Dirchlet boundary conditions also admits saddle behavior.

Recently, the existence of monotone traveling waves and the minimal wave speed was established in [36] for system (1.1) in the monostable case. Moreover, it was proven in [32] that this minimal wave speed coincides with the asymptotic speed of spread for solutions with initial functions having compact supports. The purpose of this paper is to study the existence, uniqueness and global exponential stability of traveling waves of system (1.1) with bistable nonlinearity.

Various approaches exist for proving the existence of wave solutions of parabolic equations, ranging from topoligical methods [11,19,20] to shooting methods based on Wazewski's principle ([13]). For scalar bistable evolution equations, the existence, uniqueness and global exponential stability of traveling waves are well known (see, e.g., [9, 18] for reaction-diffusion equations, [29,31] for time-delayed reaction–diffusion equations). For quasi-monotone parabolic systems with positive diffusion coefficients, monotone traveling waves were proven to exist via topological methods [33]. Also by topological methods, the existence and uniqueness of bistable traveling waves were obtained in [25] and [22], respectively, for a reaction-diffusion model of n mutualist species, in which all diffusion coefficients were assumed to be positive.

A standard approach to the local stability of traveling waves is to use the linearization at the waves under study. The stability then spilits into two steps. The first step is to prove the linear stability implies the nonlinear stability. That is, proving that the stability for the linearization implies the stability for the full nonlinear problem. The general results can be found in [2,21] and references therein. The second step is to analyze the linearized equations. All the information needed is about the spectrum of the corresponding linear operator. This is the key issue for the stability problem. For FitzHugh–Nagumo equations, the spectrum analysis [24] shows that traveling waves are stable. For quasi-monotone parabolic systems with positive diffusion coefficients, the location of the spectrum was investigated in [33,34] and references therein, and the global stability of traveling waves was obtained in [28]. In the case of positive diffusion coefficients, a general strategy for the second step was given in [1].

Evans did a series of works for an evolution system of nerve axon equations (see [14–17]), where a reaction–diffusion equation is coupled with n ordinary differential equations. In [14], he completed the first step, and the main results in [16], in fact, states that the linearized equations are stable if all spectrum points of the linear operator except for zero lie in an appropriate negative half-plane of the complex plane, and zero is a simple eigenvalue. It then follows that the local stability of bistable waves of system (1.1) reduces to the spectral analysis of the linear operator associated with the linearization at the wave profile.

The organization of this paper is as follows. In Section 2, we establish the existence of bistable waves for system (1.1) by a qualitative analysis of a three dimensional ordinary differential system. In Section 3, we use a convergence theorem for monotone semiflows to prove the global attractivity and then the uniqueness of traveliing waves (up to translations). This method seems to be new and is of its own interest. Section 4 is devoted to the global exponential stability of traveling waves. To do this, we analyze in detail the point spectrum and essential spectrum of the associated linear operator, respectively, and then use the global attractivity obtained in Section 3 and the afore-mentioned results due to Evans. A numerical simulation section completes the paper.

2. EXISTENCE OF TRAVELING WAVES

Since we are interested in the bistable case of system (1.1), throughout the whole paper we make the following assumption on the function g.

(A1) $g \in C^2(\mathbb{R}_+)$, $g(0)=0$, $g'(0) \geqslant 0$, $g'(z) > 0$, $\forall z > 0$, $\lim_{z\to\infty} g(z) = 1$, and there is a $\xi > 0$ such that $g''(z) > 0$ for $z \in (0,\xi)$ and $g''(z) < 0$ for $z > \xi$.

Mathematically, we can rescale system (1.1) and only study the rescaled system

$$\begin{aligned} \frac{\partial U_1}{\partial t}(x,t) &= \mathrm{d}\frac{\partial^2 U_1}{\partial x^2}(x,t) - U_1(x,t) + \alpha U_2(x,t), \\ \frac{\partial U_2}{\partial t}(x,t) &= -\beta U_2(x,t) + g(U_1(x,t)), \end{aligned} \tag{2.1}$$

where $\alpha = a_{12}/a_{11}^2$, $\beta = a_{22}/a_{11}$.

Let $\gamma = \beta/\alpha$. Note that the global dynamics of the cooperative system

$$\begin{aligned} \dot{U}_1(t) &= -U_1(t) + \alpha U_2(t), \\ \dot{U}_2(t) &= -\beta U_2(t) + g(U_1(t)) \end{aligned} \tag{2.2}$$

has been described in detail [6–8]. In particular, the following results are known.

Proposition 2.1. *There exists $\gamma_{crit} > 0$ such that:*

(i) For $\gamma > \gamma_{crit}$, $(0,0) \in \mathbb{R}^2$ is the only equilibrium for ODE system (2.2). It is globally asymptotically stable in the positive quadrant of $\mathbb{R}^2$;

(ii) for $\gamma = \gamma_{crit}$ or $0 < \gamma \leqslant g'(0)$ in this case of $g'(0) > 0$, system (2.2) admits a unique nontrivial equilibrium in addition to (0,0);

(iii) for $g'(0) < \gamma < \gamma_{crit}$, system (2.2) has three equilibria in the first quadrant of $\mathbb{R}^2$: $E^- = (0,0)$, $E^0 = (a, a/\alpha)$, $E^+ = (b, b/\alpha)$, where $0 < a < b$ are two positive roots of $g(x) = (\beta/\alpha)x$. In this case, E^0 is a saddle point, E^- and E^+ are stable nodes.

In order to discuss the existence of bistable waves for (2.1), i.e., traveling waves connecting two stable equilibria, we further assume $g'(0) < \gamma < \gamma_{crit}$. See Fig. 1 for an illustration of three equilibria.

Let $(U_1(x,t), U_2(x,t)) = (u_1(x+ct), u_2(x+ct))$ be a traveling wave solution of (2.1). Then the wave front profile $(u_1(\tau), u_2(\tau))$ satisfies the ODE system

$$\begin{aligned} cu_1'(\tau) &= \mathrm{d}u_1''(\tau) - u_1(\tau) + \alpha u_2(\tau), \\ cu_2'(\tau) &= -\beta u_2(\tau) + g(u_1(\tau)), \end{aligned} \tag{2.3}$$

where $'$ denotes the derivatives $\mathrm{d}/\mathrm{d}\tau$. Since we are interested in travelling wave fronts connecting E^- and E^+, we impose the asymptotic boundary

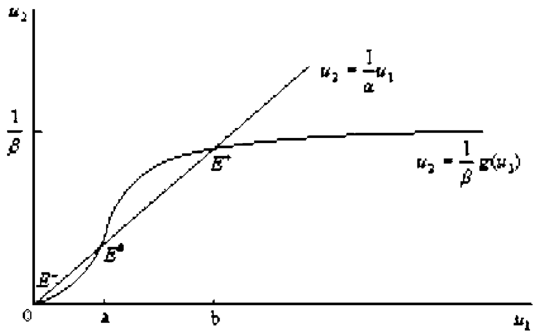

Figure 1. Illustration of E^-, E^0 and E^+.

conditions on the system

$$u_1(-\infty) = u_1'(-\infty) = u_2(-\infty) = 0. \tag{2.4}$$

We first consider the case where $c \neq 0$. By the second equation of system (2.3), we have

$$u_2(\tau) = e^{-\frac{\beta}{c}(\tau-\tau_0)}u_2(\tau_0) + \frac{1}{c}\int_{\tau_0}^{\tau} e^{-\frac{\beta}{c}(\tau-s)}g(u_1(s))\mathrm{d}s.$$

Note that, as $\tau \to -\infty$, $u_2(\tau)$ and $g(u_1(\tau))$ are bounded. By taking $\tau_0 \to -\infty$, we obtain

$$\begin{aligned} u_2(\tau) &= \frac{1}{c}\int_{-\infty}^{\tau} e^{-\frac{\beta}{c}(\tau-s)}g(u_1(s))\mathrm{d}s \\ &= \frac{1}{c}\int_{-\infty}^{0} e^{\frac{\beta}{c}s}g(u_1(\tau+s))\mathrm{d}s, \quad \forall \tau \in \mathbb{R}. \end{aligned} \tag{2.5}$$

Therefore, if $u_1(\tau)$ is increasing with

$$u_1(-\infty) = 0, \qquad u_1(+\infty) = b, \tag{2.6}$$

then $u_2(\tau)$, defined by formula (2.5), is also increasing and satisfies

$$u_2(-\infty) = 0, \qquad u_2(+\infty) = \frac{b}{\alpha}.$$

Consequently, it suffices to consider positive and increasing solutions $u_1(\tau)$ of system (2.3) subject to the boundary conditions (2.6).

Let $u_3 = u_1'$. Then system (2.3) is equivalent to

$$\begin{aligned} u_1'(\tau) &= u_3(\tau), \\ u_2'(\tau) &= \frac{1}{c}(-\beta u_2(\tau) + g(u_1(\tau))), \\ u_3'(\tau) &= \frac{1}{d}(cu_3(\tau) + u_1(\tau) - \alpha u_2(\tau)). \end{aligned} \tag{2.7}$$

Obviously, system (2.7) admits three equilibria: $(E^-, 0)$, $(E^0, 0)$ and $(E^+, 0)$. The Jacobian matrix of (2.7) is

$$J = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{c}g'(z) & -\frac{\beta}{c} & 0 \\ \frac{1}{d} & -\frac{\alpha}{d} & \frac{c}{d} \end{pmatrix}.$$

Let $f(\lambda, m) := (\lambda + \frac{\beta}{c})(-\lambda^2 + \frac{c}{d}\lambda + \frac{1}{d}) - m$. Then, at the point $(E^-, 0)$ the eigenvalues of J are given by the roots of $f(\lambda, \frac{\alpha}{cd}g'(0)) = 0$. Since $f(\lambda, \beta/cd)$ admits three real zero points: $\lambda_1 < 0$, $\lambda_2 = 0$, $\lambda_3 > 0$, and $0 \leqslant g'(0) < \beta/\alpha$, it follows that at $(E^-, 0)$, J admits a positive eigenvalue $\lambda(c)$ and two negative eigenvalues. Therefore, system (2.7) has an one-dimensional unstable manifold corresponding to $\lambda(c)$ at (0, 0, 0). Denote by $\mathcal{U}_c$ this manifold. Note that $(1, g'(0)/(\beta + c\lambda(c), \lambda(c)))$ is an eigenvector correspondinig to $\lambda(c)$. It is easy to prove the following lemma for the solution on $\mathcal{U}_c$ (see, e.g., [10]).

Lemma 2.1. *Assume that $c \neq 0$. Then system (2.3) and (2.4) has exactly one positive solution on $\mathcal{U}_c$ (up to translations). For sufficiently large negative τ, this solution satisfies*

$$u_1'(\tau) = u_3(\tau) = \lambda(c)u_1(\tau) + O(u_1(\tau)), \qquad u_2(\tau) = \frac{g'(0)}{\beta + c\lambda(c)}u_1(\tau) + O(u_1(\tau)).$$

Remark 2.1. If $g'(0) = 0$, we assume that $g''(0) \neq 0$. Then $u_2(\tau)$ Lemma 2.1 can be approximated by

$$u_2(\tau) = \frac{g''(0)}{2\beta + 4c\lambda(c)}u_1(\tau) + O(u_1^2(\tau)) \quad (\tau \to -\infty).$$

In the case where $c = 0$, system (2.3) is equivalent to

$$\mathrm{d}y_1''(\tau) - y_1(\tau) + \frac{\alpha}{\beta}g(y_1(\tau)) = 0$$

$$y_1'(\tau) = y_3(\tau), \qquad y_3'(\tau) = \frac{1}{\mathrm{d}}\left(y_1(\tau) - \frac{\alpha}{\beta}g(y_1(\tau))\right) \tag{2.8}$$

with boundary conditions $y_1(-\infty)=y_3(-\infty)=0$. In what follows, we are only interested in positive and increasing solutions $u_1(\tau)$ and $y_1(\tau)$ of (2.3) and (2.8), respectively. As long as $y_1'(\tau)\geqslant 0$, for the trajecatory $\Psi(\eta):=y_3(y_1^{-1}(\eta))$ we have the following graph equation in (y_1,y_3) phase space

$$\dot{\Psi}(\eta)=\frac{\eta-\frac{\alpha}{\beta}g(\eta)}{d\Psi(\eta)} \quad \text{for} \quad \eta>0. \tag{2.9}$$

In the case where $c\neq 0$, as long as $u_1'(\tau)\geqslant 0$, for $V(\eta):=u_1'(u_1^{-1}(\eta))$ $(\eta=u_1(\tau))$ we have

$$\dot{V}(\eta)=\frac{c}{d}+\frac{\eta-\alpha u_2(\tau)}{\mathrm{d}V(\eta)} \quad \text{for} \quad \eta>0, \tag{2.10}$$

where $u_2(\tau)=u_2(u_1^{-1}(\eta))$. For the solutions of (2.10) associated with the trajectory for (2.3) and (2.4), the boundary conditions (2.4) and Lemma 2.1 provide

$$V(0+)=0, \quad \dot{V}(0+)=\lambda(c). \tag{2.11}$$

Our proofs involve continuous "switching" between solutions for the graph equation (2.10) and the original system (2.3) or (2.7). So we first give the following lemma on some general properties of trajectories $V(\eta)$ with (2.11), which will be frequently used. Let $u(\tau)=(u_1(\tau),(u_2(\tau),(u_3(\tau))$ be the solution of system (2.7) associated with $V(\eta)$.

Lemma 2.2. *Let $c>0$. Then the following statements hold.*

(i) *$V(\eta)>0$, and $\dot{V}(\eta)\geqslant\frac{c}{d}>0$ for $\eta\in(0,a]$.*
(ii) *Let $\bar{\eta}=\inf\{\eta\in(0,b]:V(\eta)=0\}$. Then $\bar{\eta}>a$, and $\bar{\eta}<b$ implies that $\lim\limits_{n\nearrow\bar{\eta}}\frac{V(\eta)}{\eta-\bar{\eta}}=-\infty$.*

Proof. (i) As long as $V(\eta)$ is well defined (i.e., $u_1'(\tau)=u_3(\tau)\geqslant 0$), it follows from (2.5) that $u_2(\tau)\leqslant(1/\beta)g(u_1(\tau))$ for $u_1(\tau)\in(0,b]$. Therfeore, for $u_3(\tau)>0$ and $u_1(\tau)\in(0,a]$, we have $u_2(\tau)\leqslant(1/\beta)g(u_1(\tau))\leqslant(1/\alpha)u_1(\tau)$ (see Fig. 1). We claim that $u_3'(\tau)>0$ as long as $u_1(\tau)\in(0,a]$. Indeed, Lemma 2.1 implies that $u_3(\tau)>0$ for small positive $u_1(\tau)$. It follows that for small $u_1(\tau)$ there holds

$$u_3'(\tau)=\frac{1}{\mathrm{d}}(cu_3(\tau)+u_1(\tau)-\alpha u_2(\tau))\geqslant\frac{1}{\mathrm{d}}cu_3(\tau)>0.$$

Suppose, by contradiction, that $\tau_0\in\mathbb{R}$ is the first point such that $u_3'(\tau_0)=0$ and $u_1(\tau_0)\in(0,a]$. Then

$$\frac{1}{\mathrm{d}}(cu_3(\tau_0)+u_1(\tau_0)-\alpha u_2(\tau_0))=0 \quad \text{and} \quad u_3(\tau_0)>0.$$

Therefore $u_1(\tau_0) < \alpha u_2(\tau_0)$, and hence $u_1(\tau_0) > a$, a contradiction. It then folows that $u_3(\tau) > 0$ and $\alpha u_2(\tau) \leqslant u_1(\tau)$ as long as $u_1(\tau) \in (0, a]$. Thus, for $\eta \in (0, a]$, we have $V(\eta) > 0$ and $\dot{V}(\eta) = (c/d) + (u_1(\tau) - \alpha u_2(\tau))/\mathrm{d}V(\eta) \geqslant (c/d) > 0$, where $\tau = u_1^{-1}(\eta)$.

(ii) Clearly, $\bar{\eta} > a$. Suppose that $\bar{\eta} < b$. Then $u_3'(\bar{\tau}) = 0$ and $u_3'(\bar{\tau}) \leqslant 0$, where $\bar{\tau} = u_1^{-1}(\bar{\eta})$. We claim that $u_3'(\bar{\tau}) < 0$. Suppose, by contradiction, that $u_3'(\bar{\tau}) = 0$. Then $u_1(\bar{\tau}) - \alpha u_2(\bar{\tau}) = 0$, and $\bar{\tau} < +\infty$. Moreover, we can choose a small $\varepsilon > 0$ such that for $\tau \in (\bar{\tau} - \varepsilon, \bar{\tau}), u_3'(\tau) \leqslant 0$ and $u_3(\tau) > 0$, and hence, $u_1(\tau) - \alpha u_2(\tau) < 0$. Using (2.5) and the fact that $u_1'(\tau) > 0$ for $\tau < \bar{\tau}$, we then have

$$
\begin{aligned}
0 \leqslant u_1'(\bar{\tau}) - \alpha u_2'(\bar{\tau}) &= u_3(\bar{\tau}) - \frac{\alpha}{c}\Big(- \beta u_2(\bar{\tau}) + g(u_1(\bar{\tau}))\Big) \\
&= -\frac{\alpha}{c}\Big(- \beta u_2(\bar{\tau}) + g(u_1(\bar{\tau}))\Big) < 0,
\end{aligned}
$$

a contradiction. Thus $u_3'(\bar{\tau}) < 0$, and hence

$$
\lim_{\eta \nearrow \bar{\eta}} \frac{V(\eta)}{\eta - \bar{\eta}} = \lim_{\tau \nearrow \bar{\tau}} \frac{u_3'(\tau)}{u_1'(\tau)} = \lim_{\tau \nearrow \bar{\tau}} \frac{u_3'(\tau)}{u_3(\tau)} = -\infty.
$$

□

Assume that $\int_0^b ((\alpha/\beta) g(z) - z)\mathrm{d}z > 0$. Let

$$
N_k = \{(\eta, \zeta) \in \mathbb{R}^2 : \zeta^2/2 + F(\eta) = k\} \backslash \{(0,0), (a,0), (b,0)\},
$$

where $F(\eta) = (1/d)\int_0^\eta ((\alpha/\beta) g(z) - z)\mathrm{d}z$. Since $z > (\alpha/\beta) g(z)$ for $z \in (0, a)$ in the case of $g'(0) < \gamma < \gamma_{crit}$ (see Fig. 1), $k_0 = F(a) < 0$. Note that N_k are exactly the trajectories of solutions to system (2.8), and $k \geqslant 0$ gives exactly the trajectories intersecting ζ-axis (see Fig. 2). For $k \geqslant 0$, we define $N_k^+ = N_k \cap \{(\eta, \zeta) \in \mathbb{R}^2 : \zeta > 0\}$ and denote by $\Psi_k(\eta)$, the solution of Eq. (2.9) in N_k^+. Let $V_c(\eta)$ be the solution of Eqs. (2.10) and (2.11) with the velocity of c and $u_c(\tau) = (u_1(\tau), u_2(\tau), u_3(\tau))$ be the solution of system (2.7) with (2.4) corresponding to $V_c(\eta)$. Then we have the following result on the relationship between Ψ_k and V_c or u_c.

Lemma 2.3. *For $c > 0$ and $u_3(\tau) > 0, (u_1(\tau), u_3(\tau))$ crosses through increasing level sets N_k^+ with increasing τ when $u_1(\tau) \in (0, b)$. That is, $V_c(\eta)$ intersects a level set N_k^+ at most once for $\eta \in (0, b)$. Furthermore, at the point of intersection $(\eta_k, V_c(\eta_k)) = (\eta_k, \Psi_k(\eta_k))$, there holds $V_c'(\eta_k) \geqslant (c/d) + \Psi_k'(\eta_k)$.*

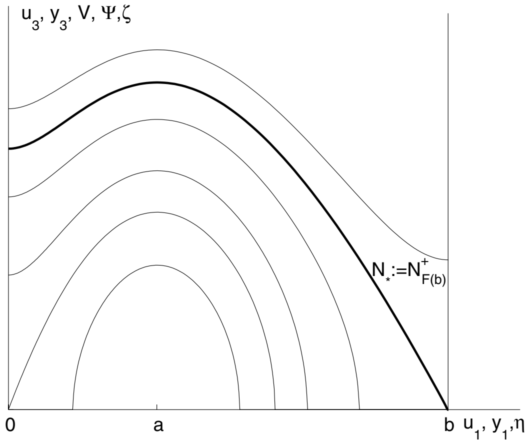

Figure 2. Phase portrait of (2.8).

Proof. Note that $u_2(\tau) \leqslant (1/\beta)g(u_1(\tau))$ whenever $u_1(\tau) \in (0,b)$. We then have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}k = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{2}u_3^2(\tau) + F(u_1(\tau))\right) = u_3(\tau)u_3'(\tau) + \frac{1}{\mathrm{d}}\left(\frac{\alpha}{\beta}g(u_1(\tau)) - u_1(\tau)\right)u_1'(\tau)$$
$$= \frac{c}{\mathrm{d}}u_3^2(\tau) + \frac{1}{\mathrm{d}}u_3(\tau)\left(\frac{\alpha}{\beta}g(u_1(\tau)) - \alpha u_2(\tau)\right) > 0.$$

This proves the first part of the result. The second part follows from a direct computation

$$V_c'(\eta_k) = \frac{c}{d} + \frac{\eta_k - \alpha u_2(\tau_k)}{V_c(\eta_k)} \geqslant \frac{c}{d} + \frac{\eta_k - \frac{\alpha}{\beta}g(\eta_k)}{\Psi_k(\eta_k)} = \frac{c}{d} + \Psi_k'(\eta_k).$$

□

Now we are ready to prove the main result of this section.

Theorem 2.1. *Assume that $g'(0) < \gamma < \gamma_{crit}$. Then there exists a wave speed c^* such that system (2.1) has a nontrivial strictly monotone wave solution connecting E^- and E^+, and c^* has the same sign as $\int_0^b ((\alpha/\beta)g(z) - z)\,\mathrm{d}z$. Moreover, $c^* = 0$ iff the integral vanishes.*

Proof. Without loss of generality, we assume that

$$\int_0^b \left(\frac{\alpha}{\beta}g(z) - z\right)\mathrm{d}z \geqslant 0. \tag{2.12}$$

Otherwise, by the change of varibles $V_1 = b - U_1$, $V_2 = (1/\alpha)b - U_2$, we can transform the original system (2.1) into

$$\frac{\partial}{\partial t}V_1(x,t) = \mathrm{d}\frac{\partial^2}{\partial x^2}V_1(x,t) - V_1(x,t) + \alpha V_2(x,t),$$
$$\frac{\partial}{\partial t}V_2(x,t) = G(V_1(x,t)) - \beta V_2(x,t),$$

where $G(z) = g(b) - g(b-z)$. since $g(b) = (\beta/\alpha)b$, $G(z)$ satisfies assumption (A1)on [0,b) and (2.12).

If $c^* = 0$, the heteroclinic orbit of system (2.8) implies that the integral (2.12) vanishes. Conversely, if the integral vanishes, there is a nontrival wave solution of the ODE system (2.3) with velocity $c = 0$. Therefore, we restrict ourselves to the positive integral.

Let $N_* := N^+_{F(b)}$ be the level curve through the critical point $(b, 0)$ and $\Psi_*(\eta) := \Psi_{F(b)}(\eta)$ be the corresponding solution. Define

$E = \{c > 0 : V_c$ and N_* intersect at $(\eta_c, V_c(\eta_c)) = (\eta_c, \Psi_*(\eta_c))$ with $\eta_c \in (0, b]\}$.

In the rest of the proof, we proceed with four steps.

Step 1: $E \neq \emptyset$. For $c > 0$ and $\eta \in (0, a]$, we have $V_c(\eta) > 0$ and $\dot{V}_c(\eta) \geqslant c/d > 0$. If $\dot{V}_c(\eta) > 0$ for $\eta \in (a, b)$, then $V_c(\eta)$ must intersect with N_* on (0,b]. Suppose that $\dot{V}_c(\eta_0) = 0$ for $\eta_0 \in (a, b)$. Then $V_c(\eta_0) = (1/c)(\alpha u_2(\tau_0) - u_1(\tau_0)) > V_c(a) - V_c(0) = a\dot{V}(\eta_1)$, where $\eta_1 \in (0, a)$, $\tau_0 = u_1^{-1}(\eta_0)$. We then have $\alpha u_2(\tau_0) - u_1(\tau_0) > (c^2/d)a$. Note that $0 < u_2(\tau_0) \leqslant (b/\alpha)$. Thus, whenever, $c^2 \geqslant (bd/a)$, we have $\dot{V}_c(\eta) > 0$ for $\eta \in (a, b)$. Then $V_c(\eta)$ intersects N_*. Thus, $E \neq \emptyset$.

Step 2: $\underline{c} = \inf E > 0$. Let $m > 0$ be a constant. Consider the line $V = -m(\eta - b)(\eta \in [0, b])$. If $V_c(\eta)$ intersect with this line, then at the intersection we have

$$\dot{V}_c(\eta) = \frac{c}{d} - \frac{\eta - \alpha u_2}{md(\eta - b)} \leqslant \frac{c}{d} + \frac{b - \eta}{md(\eta - b)} = \frac{1}{d}\left(c - \frac{1}{m}\right).$$

For any sufficiently small $c > 0$, we can choose $m \in (0, -c/2 + \sqrt{1 + c^2/4})$ such that $c - 1/m < -m$. Then we must have $V_c(\eta_c) = 0$ for some $\eta_c \in (a, b]$. Thus, by Lemma 2.3, V_c does not intersect with N_* on (0,b]. Therefore, $\underline{c} > 0$.

Step 3: $\underline{c} \in E$. Suppose, by contradiction , that $V_{\underline{c}}(\eta)$ does not intersect with N_*. Then Lemmas 2.2 and 2.3 imply that $V_{\underline{c}}(\bar{\eta}) = 0$ for $\bar{\eta} \in (a, b]$. If $\bar{\eta} = b$, we are done.

Assume that $\bar{\eta} < b$. Let P_1 be a small plane in a small neighborhood of $(0,0,0)$ in (u_1,u_2,u_3) phase space, which is normal to the eigenvector $(1, m(\underline{c}), \lambda(\underline{c}))$ corresponding to the eigenvector $\lambda(\underline{c})$, where $m(\underline{c}) = g'(0)/(\beta + c\lambda(\underline{c}))$. By Lemma 2.1, the trajectory $u_{\underline{c}}(\tau)$ transversely intersects with P_1 at $I_{\underline{c}}$. By the local continous dependence of $\mathcal{U}_{\underline{c}}$ on $\underline{c}$, for all c in a small neighborhood of $\underline{c}, u_c(\tau)$ transversely crosses through P_1 at I_c and $\lim_{c \to \underline{c}} I_c = I_{\underline{c}}$ Without loss of generality, we assume that $u_c(0) = I_c, u_{\underline{c}}(0) = I_{\underline{c}}$. Let $P_2 = \{(u_1,u_2,u_3) \in \mathbb{R}^3 : u_1, u_2 > 0, u_3 = 0\}$. Then, Lemma 2.2 implies that $u_{\underline{c}}(\tau)$ transversely intersects P_2 at $(\bar{\eta}, u_2(u_1^{-1}(\bar{\eta})), 0)$. By the continuous dependence of solutions on parameters and initial values, for all I_c in a sufficiently small neighborhood of $I_{\underline{c}}, u_c(\tau)$ transversely intersects P_2. Thus, we can choose a $c > \underline{c}$ such that $u_c(\tau)$ intersects P_2. That is, $V_c(\eta) = 0$ for some $\eta \in (a,b)$. By Lemma 2.3, $V_c(\eta)$ has no intersection points with N_*. Hence $\underline{c} < c \notin E$, which contradicts the definition of $\underline{c}$. Therefore, $V_{\underline{c}}$ does intersect with N_*. That is, $\underline{c} \in E$.

Step 4. $\eta_{\underline{c}} = b$. Suppose that $\eta_{\underline{c}} < b$. Let $P_3 = \{(u_1,u_2,u_3) \in \mathbb{R}^3 : (u_1,u_3) \in N_*, u_2 > 0\}$. Then by Lemma 2.3, $u_{\underline{c}}(\tau)$ transversely intersects P_3 at $(\eta_{\underline{c}}, u_2(u_1^{-1}(\eta_{\underline{c}})), V_{\underline{c}}(\eta_{\underline{c}}))$. By the same argument as in step 2, we obtain that, as $c \to \underline{c}, u_c(\tau)$ transversely intersects P_3 at $(\eta_c, u_2(u_1^{-1}(\eta_c)), V_c(\eta_c))$, and $\eta_c \to \eta_{\underline{c}} < b$. It follows that there exists a $\delta > 0$ such that $\underline{c} - \delta \in E$, which contradicts the definition of $\underline{c}$. □

Remark 2.2. Note that

$$\int_0^b \left(\frac{\alpha}{\beta} g(z) - z\right) \mathrm{d}z = \alpha \left[\int_a^b \left(\frac{1}{\beta} g(z) - \frac{1}{\alpha} z\right) \mathrm{d}z - \int_0^a \left(\frac{1}{\alpha} z - \frac{1}{\beta} g(z)\right) \mathrm{d}z\right].$$

By Theorem 2.1, it then foliows that exists $\bar{\gamma} \in (g'(0), \gamma_{\mathrm{crit}})$ such that $c^* > 0$ if $\gamma \in (g\prime(0), \bar{\gamma})$, and $c^* < 0$ if $\gamma \in (\bar{\gamma}, \gamma_{\mathrm{crit}})$.

3. ATTRACTIVITY AND UNIQUENESS

In this section, we discuss the global attractivity and uniqueness of travelling waves of system (2.1). For convenience, in the rest of the paper we considr a more general quasi-monotone system

$$\frac{\partial U_1}{\partial t}(x,t) = \mathrm{d}\frac{\partial^2 U_1}{\partial x^2}(x,t) + F^1(U_1(x,t), U_2(x,t)),$$
$$\frac{\partial U_2}{\partial t}(x,t) = -\beta U_2(x,t) + g(U_1(x,t)) := F^2(U_1(x,t), U_2(x,t)). \quad (3.1)$$

Assume that

(A2) There exists $l > 0$ such that $F^1 \in C^2((-l,\infty)^2, \mathbb{R})$, and $(\partial/\partial u_1)F^1(u_1,u_2) < 0$, $(\partial/\partial u_2)F^1(u_1,u_2) > 0$ for $(u_1,u_2) \in (-l,\infty)^2$.

(A3) $F^1(0,0) = 0$, and for any $l_2 \geqslant 1/\beta$, there exists $l_1 > 0$ such that $F^1(l_1,l_2) < 0$.

Without loss of generality, we may assume that the function g admits a smooth extension defined on $(-l,\infty)$ with $g'(z) \geqslant 0$ for $z \in (-l,0)$. In what follows, we use notations

$$F^i_j(u_1,u_2) := \frac{\partial}{\partial u_j}F^i(u_1,u_2), \quad F^i_{jk}(u_1,u_2) := \frac{\partial^2}{\partial u_j \partial u_k}F^i(u_1,u_2), \quad 1 \leqslant i,j,k \leqslant 2.$$

Consider the ODE system

$$w_1'(t) = F^1(w_1(t),\ w_2(t)),$$
$$w_2'(t) = F^2(w_1(t),\ w_2(t)). \quad (3.2)$$

Because of our assumptions on F^1 and g, system (3.2) is cooperative on $\mathbb{R}^2_+$. Hence the comparison principle implies that every solution to (3.2) with nonnegative initial values remains nonnagative. By the standard compapison arguments, it easily follows that solutions of (3.2) on $\mathbb{R}^2_+$ are uniformly bounded and ultimately bounded. Thus, each solution of (3.2) with nonnagative initial values exists globally on $[0,\infty)$, and the solution semiflow of (3.2) is compact, point dissipative, and monotone on $\mathbb{R}^2_+$.

Obviously, E^- is an equilibrium of (3.2). We further assume that (3.2) admits two nonnagative equilibrim in $\mathbb{R}^2_+$. With a little abuse of notations, we denote them by E^0 and E^+. Furthermore, suppose that $E^- \ll E^0 \ll E^+$, and $E^\pm$ are stable nodes and E^0 is a saddle point, where "$\ll$" means that components of the two vectors satisfy "$<$". Define $[E^-,E^0] = \{w \in \mathbb{R}^2_+ : E^- \leqslant w \leqslant E^0\}$ and $[E^0,\infty) = \{w \in \mathbb{R}^2_+ : E^0 \leqslant w\}$. By the Dancer–Hess connecting orbit lemma (see [12, Proposition 1]) and [30, Therom 2.3.2], as applied to $[E^-,E^0]$ and $[E^0,\infty)$, respectively, it follows that $\lim_{t\to\infty} w(t,w_0) = E^-$ for $w_0 \in [E^-,E^0]\backslash\{E^0\}$ and $\lim_{t\to\infty} w(t,w_0) = E^+$ for $w_0 \in [E^0,\infty)\backslash\{E^0\}$, where $w(t,w_0)$ is the solution to (3.2) with $w(0,w_0) = w_0 \in \mathbb{R}^2_+$.

Let $\mathbb{X} := BUC(\mathbb{R}, \mathbb{R}^2)$ be the Banach space of all bounded and uniformly continouous functions from $\mathbb{R}$ to $\mathbb{R}^2$ with the usual supreme norm. Let $\mathbb{X}_+ = \{(\psi_1, \psi_2) \in \mathbb{X} : \psi_i(x) \geqslant 0, \forall x \in \mathbb{R}, i = 1, 2\}$. It is easy to see that $\mathbb{X}_+$ is a closed cone of $\mathbb{X}$ and its induced partial ordering makes $\mathbb{X}$ into a Banach lattice. For any $\psi^1 = (\psi_1^1, \psi_2^1), \psi^2 = (\psi_1^2, \psi_2^2) \in \mathbb{X}$, we write $\psi^1 \leqslant_{\mathbb{X}} \psi^2$ if $\psi^2 - \psi^1 \in \mathbb{X}_+$, $\psi^1 <_{\mathbb{X}} \psi^2$ if $\psi^2 - \psi^1 \in \mathbb{X}_+ \backslash \{0\}$, $\psi^1 \ll_{\mathbb{X}} \psi^2$ if $\psi^2 - \psi^1 \in int(\mathbb{X}_+)$. For $\psi^1, \psi^2 \in \mathbb{X}$ with $\psi^1 \leqslant_{\mathbb{X}} \psi^2$, let $[\psi^1, \psi^2]_{\mathbb{X}} = \{\psi \in \mathbb{X} : \psi^1 \leqslant_{\mathbb{X}} \psi \leqslant_{\mathbb{X}} \psi^2\}$.

To prove the global attractivity and uniqueness of travelling waves, we need a series of lemmas.

Lemma 3.1. *For any $\psi \in \mathbb{X}_+$, system (3.1) has a unique, bounded and nonnegative solution $U(x, t, \psi)$ with $U(\cdot, 0, \psi) = \psi$, and the solution semiflow of (3.1) is monotone. Moreover, $U(x, t, \psi^1) \ll U(x, t, \psi^2)$ for $t > 0$ and $x \in \mathbb{R}$ whenever $\psi^1, \psi^2 \in \mathbb{X}_+$ with $\psi^1 <_{\mathbb{X}} \psi^2$.*

Proof. Let $T_1(t)$ be the analytic semigroup on $BUC(\mathbb{R}, \mathbb{R})$ generated by $\partial u/\partial t = \partial^2 u/\partial x^2$, and $T_2(t)\psi_2 = e^{-\beta t}\psi_2, \forall \psi_2 \in BUC(\mathbb{R}, \mathbb{R})$. Clearly, $T(t) = (T_1(t), T_2(t))$ is a linear semigroup on $\mathbb{X}$. Let $B(\psi)(x) = (F^1(\psi_1(x), \psi_2(x)), g(\psi_1(x))), \forall \psi = (\psi_1, \psi_2) \in \mathbb{X}_+$. Then a mild solution $U(t)$ of (3.1) is defined as a solution to the integral equation

$$U(t) = T(t)U(0) + \int_0^t T(t-s)B(U(s))\mathrm{d}s.$$

It is easy to check the quasi-monotonicity of $B(\psi)$. By [26, corollary 5] (taking delay as zero), it then follows that for any $\psi \in \mathbb{X}_+$, system (3.1) has a unique nonnegative and noncontinuable mild solution $U(x, t, \psi)$ satisfying $U(\cdot, 0, \psi) = \psi$. Moreover, by a semigroup theory argument given in the proof of [26, Theorem 1], it follows that $U(x, t, \psi)$ is a classical solution for $t > 0$. Note that [26, Corollary 5] also implies that the comparison principle holds for system (3.1). By the comparison argument, solutions of (3.1) on $\mathbb{X}_+$ are uniformly bounded. Therefore, system (3.1) defines a monotone solution semiflow on $\mathbb{X}_+$.

Suppose that $\psi^1, \psi^2 \in \mathbb{X}_+$ with $\psi^1 <_{\mathbb{X}} \psi^2$. Then $U(x, t, \psi^i) \geqslant 0, \forall x \in \mathbb{R}, t \geqslant 0$. Let $U(x, t) = U(x, t, \psi^2) - U(x, t, \psi^1)$. Then $U(x, t) \geqslant 0, \forall x \in \mathbb{R}, t \geqslant 0$, and $U(\cdot, 0) \not\equiv 0$. Note that the first component $U_1(x, t)$ of $U(x, t)$ satisfies

$$U_{1,t} = \mathrm{d}U_{1,xx} + \sum_{i=1}^{2} U_i \int_0^1 F_i^1(sU(x, t, \psi^2) + (1-s)U(x, t, \psi^1))\mathrm{d}s \quad (3.3)$$

$$\geqslant \mathrm{d}U_{1,xx} + U_1 \int_0^1 F_1^1(sU(x, t, \psi^2) + (1-s)U(x, t, \psi^1))\mathrm{d}s, \quad (3.4)$$

and the second component $U_2(x,t)$ of $U(x,t)$ satisfies

$$U_{2,t} = -\beta U_2 + A(x,t)U_1,$$

where $A(x,t) = \int_0^1 g'(sU_1(x,t,\psi^2) + (1-s)U_1(x,t,\psi^1))\mathrm{d}s$, and $U_1(x,t,\psi^1)$, $U_1(x,t,\psi^2)$ are the first components of $U(x,t,\psi^1)$ and $U(x,t,\psi^2)$, respectively. It then follows that

$$U_2(x,t) = e^{-\beta t}U_2(x,0) + \int_0^t e^{-\beta(t-s)}A(x,s)U_1(x,s)\mathrm{d}s. \tag{3.5}$$

In the case where $U_1(\cdot,0) \not\equiv 0$, the strict positivety theorem [33, Theorem 5.5.4] and inequality (3.4) imply that $U_1(x,t) > 0, \forall x \in \mathbb{R}, t > 0$. Since $g'(z) > 0$ for $z > 0$, (3.5) implies $U_2(x,t) > 0, \forall x \in \mathbb{R}, t > 0$. Thus, $U(x,t,\psi^1) \ll U(x,t,\psi^2)$ for $x \in \mathbb{R}, t > 0$.

In the case where $U_2(\cdot,0) \not\equiv 0$, it follows from (3.5) that $U_2(\cdot,t) \not\equiv 0$ for $t \geqslant 0$. Since $F_2^1 > 0$ on $\mathbb{R}_+^2$. the equality (3.3) implies that $U_1(.,t) \not\equiv 0$ for $t > 0$, and hence by the inequality (3.4) and [33, Theorem 1.4.5], we must have $U_1(x,t) > 0$, $\forall x \in \mathbb{R}, t > 0$. Therefore, it follows from (3.5) that $U_2(x,t) > 0$, $\forall x \in \mathbb{R}, t > 0$. Thus $U(x,t,\psi^1) \ll U(x,t,\psi^2)$, $\forall x \in \mathbb{R}, t > 0$. □

In view of Section 2, we suppose that $\varphi(x-ct) = (\varphi_1(x-ct), \varphi_2(x-ct)$ is a strictly increasing travelling wave solution of (3.1) connecting E^- and E^+. By the moving coordinate $z = x - ct$, we transform (3.1) into the following system:

$$\begin{aligned} u_{1,t}(z,t) &= cu_{1,z}(z,t) + \mathrm{d}u_{1,zz}(z,t) + F^1(u_1(z,t), u_2(z,t)), \\ u_{2,t}(z,t) &= cu_{2,z}(z,t) + F^2(u_1(z,t), u_2(z,t)). \end{aligned} \tag{3.6}$$

Then $\varphi(z)$ is an equilibrium solution of system (3.6). In what follows, we denote by $u(z,t,\psi) = (u_1(z,t), u_2(z,t))$ the solution of system (3.6) with $u(\cdot,0,\psi) = \psi \in \mathbb{X}_+$. Clearly, the solution $U(x,t,\psi)$ of (3.1) with initial value ψ is given by $U(x,t,\psi) = u(x-ct,t,\psi)$. As noted before, the comparison principle holds for (3.1) and hence for (3.6). For convenience, we set

$$\begin{aligned} N_1(u_1,u_2) &:= u_{1,t}(z,t) - cu_{1,z}(z,t) - du_{1,zz}(z,t) - F^1(u_1(z,t), u_2(z,t)) = 0, \\ N_2(u_1,u_2) &:= u_{2,t}(z,t) - cu_{2,z}(z,t) - F^2(u_1(z,t), u_2(z,t)) = 0. \end{aligned}$$

Lemma 3.2. *If $\psi = (\psi_1, \psi_2) \in \mathbb{X}_+$ satisfies*

$$\limsup_{\xi \to -\infty} \psi(\xi) \ll E^0 \ll \liminf_{\xi \to \infty} \psi(\xi), \tag{3.7}$$

then, for any $\varepsilon > 0$, there exist $\tilde{z} = \tilde{z}(\epsilon, \psi) > 0$ and a large time $\tilde{t} = \tilde{t}(\varepsilon, \psi)$ such that $\varphi(z - \tilde{z}) - \varepsilon \leqslant u(z, \tilde{t}, \psi) \leqslant \psi(z + \tilde{z}) + \varepsilon$.

Proof. Without loss of generality, we assume that $\psi(\xi) \leqslant l_1, \forall \xi \in \mathbb{R}$ and $\psi(\xi) \leqslant l_2, \forall \xi \leqslant 0$, where $l_1, l_2 \in \mathbb{R}^2_+, l_1 \geqslant E^+, E^- \leqslant l_2 \ll E^0$. Let $v^+(t) = (v_1^+(t), v_2^+(t)) := w(t, 2l_1 - l_2)$ and $v^-(t) := w(t, l_2)$ be the solution of the reaction system (3.2) with $v^+(0) = 2l_1 - l_2, v^-(0) = l_2$. Define $\varsigma(s) = (1/2)(1 + \tanh(s/2))$. Then $\varsigma' = \varsigma(1-\varsigma), \varsigma'' = \varsigma'(1-2\varsigma)$. Let

$$\bar{c} = c + d + \sup\{\frac{(v_1^+(t) - v_1^-(t))^2}{v_i^+(t) - v_i^-(t)}|F^i_{11}(\theta)| + \frac{(v_2^+(t) - v_2^-(t))^2}{v_i^+(t) - v_i^-(t)}|F^i_{22}(\theta)|$$
$$+ 2(v_j^+(t) - v_j^-(t))|F^i_{12}(\theta)| : t \in [0, +\infty), \theta \in (v^-(t), v^+(t)), 1 \leqslant i \neq j \leqslant 2\}.$$

Let $\tilde{c} \geqslant \bar{c}$ be a fixed number, and define the function

$$v(z,t) = v^+(t)\zeta(z + \tilde{c}t) + v^-(t)(1 - \zeta(z + \tilde{c}t)), \quad \forall z \in \mathbb{R}, \quad t \geqslant 0.$$

It easily follows that $v(\cdot, 0) \geqslant \psi(\cdot)$. We further claim that $v(z,t)$ is a super-solution of system (3.6). Indeed, by Taylor's expansion, we have

$$\begin{aligned}\Lambda_i &:= \zeta F^i(v^+) + (1-\zeta)F^i(v^-) - F^i(\zeta v^+ (1-\zeta)v^-) \\ &= \frac{1}{2}\zeta(1-\zeta)(v_1^+ - v_1^-)^2 F^i_{11}(\theta) + \frac{1}{2}\zeta(1-\zeta)(v_2^+ - v_2^-)^2 F^i_{22}(\theta) \\ &\quad + \zeta(1-\zeta)(v_1^+ - v_1^-)(v_2^+ - v_2^-)F^i_{12}(\theta),\end{aligned}$$

where $\theta \in (v^-(t), v^+(t))$. For each $i = 1, 2$, and $(z,t) \in \mathbb{R} \times \mathbb{R}_+$, we have

$$\begin{aligned}N_i(v(z,t)) &= v_{i,t}(z,t) - cv_{i,z}(z,t) - d_i v_{i,zz}(z,t) - F^i(v(z,t)) \\ &= \zeta F^i(v^+) + (1-\zeta)F^i(v^-) - F^i(v(z,t)) \\ &\quad + \zeta(1-\zeta)[(\tilde{c} - c)(v_i^+ - v_i^-) - d_i(1-2\zeta)(v_i^+ - v_i^-)] \geqslant 0,\end{aligned}$$

where $d_1 = d, d_2 = 0$, and $v_i(z,t)$ is the i-th component of $v(z,t)$. Therefore $v(z,t)$ is a super-solution of system (3.6).

Thus, by the comparison principle we have $u(z,t,\psi) \leqslant v(z,t), \forall t \geqslant 0$. Note that $\lim_{t \to \infty} v^\pm(t) = E^\pm$. It then follows that for any $\varepsilon > 0$, there exists $\tilde{t} = \tilde{t}(\varepsilon, \psi) > 0$ and $\tilde{z} = \tilde{z}(\varepsilon, \psi) \in \mathbb{R}$ such that $u(z, \tilde{t}, \psi) \leqslant \varphi(z + \tilde{z}) + \varepsilon, \forall z \in \mathbb{R}$. A similar estimate on the lower bound of the solution completes the proof. □

Note that $E^\pm$ are stable nodes for the reaction system (3.2), i.e., the Jacobian matrixes $(F^i_j(E^\pm))$ have only negative eigenvalues. Let $A^\pm = (\mu^\pm_{ij})$ be the constant matrixes so that $F^i_j(E^\pm) < \mu^\pm_{ij}, 1 \leqslant i, j \leqslant 2$, and that $A^\pm$ are irreducible and have only negative eigenvalues. Denote by $\rho^\pm = (\rho_1^\pm, \rho_2^\pm)$ the positive eigenvectors corresponding to the principle eigenvalues of $A^\pm$. Let $\rho_1(\xi), \rho_2(\xi)$ be smooth positive functions such that $\rho(\xi) = (\rho_1(\xi), \rho_2(\xi)) \to \rho^\pm$ in $\mathbb{C}^2$-topology as $\xi \to \pm\infty$. Motivated by [27], we have the following result on super- and sub-solution for (3.6).

Lemma 3.3. *There exist positive numbers σ and ς_0 such that for any $\varsigma \geqslant \varsigma_0$, any $\hat{z} \in \mathbb{R}$, and $\varepsilon \in (0, \varepsilon_0(\varsigma))$,*

$$w^{\pm}(z,t) = \varphi(z \pm \hat{z} \pm \varsigma\varepsilon(1 - e^{-\sigma t})) \pm \varepsilon\rho(z \pm \hat{z})e^{-\sigma t}, \quad \forall z \in \mathbb{R}, \ \ t \geqslant 0,$$

are super- and sub-solutions of systems (3.6), respectively.

Proof. Clearly, there exist $\delta, k > 0$ such that

$$F^i_j(u) \leqslant \mu^{\pm}_{ij} \text{ for } \|u - E^{\pm}\| \leqslant \delta, u \in \mathbb{R}^2, \quad 1 \leqslant i, j \leqslant 2, \tag{3.8}$$

$$\sum_{j=1}^{2} \mu^{\pm}_{ij}\rho_j \leqslant -k\rho_i \qquad \text{for } \rho = (\rho_1, \rho_2) \in \mathbb{R}^2_+ \qquad \text{with } \|\rho - \rho^{\pm}\| \leqslant \delta, \ \ i = 1, 2.$$

Since $\varphi(\xi) \to E^{\pm}, \rho(\xi) \to \rho^{\pm}, \rho\prime(\xi), \rho\prime\prime(\xi) \to 0$ as $\xi \to \pm\infty$, there exist $\epsilon_1, M > 0$ such that

$$\begin{aligned} &k - c\varepsilon_1 - d\varepsilon_1 > 0, \\ &|\rho_i'(\eta)|, |\rho_i''(\eta)| \leqslant \varepsilon_1\rho_i(\eta), \quad \forall |\eta| \geqslant M - 1, \ \ i = 1, 2, \\ &\|\rho(\eta) - \rho^+\| \leqslant \delta, \quad \forall \eta \geqslant M - 1; \ \ \|\rho(\eta) - \rho^-\| \leqslant \delta, \quad \forall \eta \leqslant -M + 1, \\ &\|\varphi(\xi) + \varepsilon\rho(\eta) - E^+\| \leqslant \delta, \quad \forall \varepsilon \in (0, \varepsilon_1], \ \ \xi \geqslant M - 1, \quad \eta \geqslant M - 1, \\ &\|\varphi(\xi) + \varepsilon\rho(\eta) - E^-\| \leqslant \delta, \quad \forall \varepsilon \in (0, \varepsilon_1], \ \ \xi \leqslant -M + 1, \quad \eta \leqslant -M + 1. \end{aligned} \tag{3.9}$$

Let $B_1 > 0$ so that $\|\rho(\eta)\|, \|\rho'(\eta)\|, \|\rho''(\eta)\| \leqslant B_1$ for all $\eta \in \mathbb{R}$. Define

$$B_2 = \sup\{|F^i_j(u)| : u \in [E^- - \delta\vec{e}, E^+ + B_1\vec{e}]\}, B_3 = \inf_{\|\xi\| \leqslant M} \|\varphi'(\xi)\|.$$

Choosing $0 < \sigma \leqslant k - c\varepsilon_1 - d\varepsilon_1$, set

$$\varsigma \geqslant \varsigma_0 = \frac{B_1}{\sigma B_3}(B_2 + \sigma + c + d), \qquad \varepsilon_0 = \min\left\{\varepsilon_1, \frac{1}{\varsigma}\right\}.$$

With $q = e^{-\sigma t}$, the argument of φ and φ_i being $\xi = z + \hat{z} + \varsigma\varepsilon(1 - e^{-\sigma t})$ and ρ, ρ_i being $\eta = z + \hat{z}$, for any $\varepsilon \in (0, \varepsilon_0)$, we have

$$\begin{aligned} N_i(w^+(z,t)) &= w^+_{i,t}(z,t) - cw^+_{i,z}(z,t) - d_i w^+_{i,zz}(z,t) - F^i(w^+(z,t)) \\ &= F^i(\varphi) - F^i(\varphi + \varepsilon\rho q) + \varepsilon\varsigma\sigma q\varphi_i' - (\rho_i\sigma + c\rho_i' + d_i\rho_i'')\varepsilon q, \end{aligned}$$

where $d_1 = d, d_2 = 0$. We distinguish among three cases.

Case (i): $|\xi| \leqslant M$. Note that $F^i_j > 0$ for $i \neq j$, $F^i_j \leqslant 0$ for $i = j$, and $\varphi_i' > 0$. By the choice of ε_0, ς, and σ, we have

$$F^i(\varphi) - F^i(\varphi + \varepsilon\rho q) = -\int_0^1 \varepsilon q \left(\sum_{j=1}^{2} \rho_j F^i_j(\varphi + \varepsilon s\rho q)\right) \mathrm{d}s \geqslant -B_1 B_2 \varepsilon q$$

and hence

$$N_i(w^+(z,t)) \geqslant -B_1 B_2 \varepsilon q - B_1 \varepsilon q(\sigma + c + d_i) + B_3 \varepsilon \varsigma \sigma q \geqslant 0.$$

Case (ii): $\xi \geqslant M$. Since $\xi - \eta \leqslant \varsigma\varepsilon \leqslant 1, \xi > \eta \geqslant M-1$. Thus, by (3.9), we have

$$\|\varphi(\xi) + s\varepsilon\rho(\eta) - E^+\| \leqslant \delta, \quad \|\rho(\eta) - \rho^+\| \leqslant \delta, \quad \forall s \in (0,1).$$

Therefore, by (3.8), there holds

$$\begin{aligned} F^i(\varphi) - F^i(\varphi + \varepsilon\rho q) &= -\int_0^1 \varepsilon q \left(\sum_{j=1}^2 \rho_j F^i_j(\varphi + s\varepsilon\rho q) \right) \mathrm{d}s \\ &\geqslant -\varepsilon q \sum_{j=1}^2 \mu_{ij}\rho_j \geqslant k\varepsilon q \rho_i. \end{aligned}$$

Hence,

$$\begin{aligned} N_i(w^+(z,t)) &\geqslant k\varepsilon\rho_i q - (\rho_i\sigma + c\rho_i' + d_i\rho_i'')\varepsilon q \\ &\geqslant (k - \sigma - c\varepsilon_1 - d_i\varepsilon_1)\varepsilon q \rho_i \geqslant 0. \end{aligned}$$

Case (iii): $\xi \leqslant -M$. By an argument similar to case (ii), we have $N_i(w^+(z,t)) \geqslant 0$.

Combining Cases (i)–(iii), we have $N_i(w^+(z,t)) \geqslant 0.$ for all $\varepsilon \in (0, \varepsilon_0)$ and $t \geqslant 0$. Thus $w^+(z,t)$ is a super-solution of system (3.6). By a similar argument, we can prove that $w^-(z,t)$ is a sub-solution. □

Lemma 3.4. *The wave profile $\varphi(z)$ is a Liapunov stable equilibrium of (3.6).*

Proof. Let ε_0 and $w^\pm(z,t,\varepsilon)$ be defined as in Lemma 3.3 with $\hat{z} = 0$ and $\varsigma = \varsigma_0$. It then follows that there exists $K > 0$, independent of ε, such that $\|w^\pm(z,t,\varepsilon) - \varphi(z)\| < K\varepsilon, \forall z \in \mathbb{R}, t \geqslant 0, \varepsilon \in (0,\varepsilon_0)$. For any $\varepsilon \in (0,\varepsilon_0)$, let $\delta = \varepsilon \inf_{z\in\mathbb{R}} \rho(z)$. Thus, for any given $\|\psi - \varphi\| < \delta$, we have

$$w^-(z,0,\varepsilon) = \varphi(z) - \varepsilon\rho(z) \leqslant \psi(z) \leqslant \varphi(z) + \varepsilon\rho(z) = w^+(z,0,\varepsilon), \quad \forall z \in \mathbb{R}.$$

Then the comparison principle implies that $w^-(z,t,\varepsilon) \leqslant u(z,t,\psi) \leqslant w^+(z,t,\varepsilon)$, $\forall z \in \mathbb{R}, t \geqslant 0$, and hence $\|u(\cdot,t,\psi) - \varphi(\cdot)\| < K\varepsilon, \forall t \geqslant 0$. □

To prove the attractivity and uniqueness of traveling waves, we need the following convergence result for monotone semiflows, which comes from [35, Theorem 2.2.4].

Lemma 3.5. *Let U be a closed convex subset of an ordered Banach space χ, and $\Phi(t): U \to U$ be a monotone semiflow. Assume that there exists a monotone homeomorphism h from [0, 1] onto a subset of U such that:*

(1) For each $s \in [0,1], h(s)$ is a stable equilibrium for $\Phi(t): U \to U$;
(2) each orbit of $\Phi(t)$ in $[h(0), h(1)]\chi$ is precompact;
(3) one of the following two properties holds:

(3a) if $h(s_0) <_\chi \omega(\phi)$ for some $s_0 \in [0,1)$ and $\phi \in [h(0), h(1)]_\chi$, then there exists $s_1 \in (s_0, 1)$ such that $h(s_1) \leqslant_\chi \omega(\phi)$;
(3b) if $\omega(\phi) <_\chi h(r_1)$ for some $r_1 \in [0,1)$ and $\phi \in [h(0), h(1)]_\chi$, then there exists $r_0 \in (0, r_1)$ such that $\omega(\phi) \leqslant_\chi h(r_0)$.

Then for any precompact orbit $\gamma^+(\phi_0)$ of $\Phi(t)$ in U with $\omega(\phi_0) \cap [h(0), h(1)]_\chi \neq \emptyset$, there exists $s^* \in [0,1]$ such that $\omega(\phi_0) = h(s^*)$.

Now we are in a position to prove the main result of this section.

Theorem 3.1. *Let $\varphi(x-ct)$ be a monotone traveling wave solution of system (3.1) and $U(x,t,\psi)$ be the solution of (3.1) with $U(\cdot, 0, \psi) = \psi(\cdot) \in \mathbb{X}_+$. Then for any $\psi \in \mathbb{X}_+$, satisfying (3.7), there exists $s_\psi \in \mathbb{R}$ such that $\lim_{t\to+\infty} \|U(x,t,\psi) - \varphi(x - ct + s_\psi)\| = 0$ uniformly for $x \in \mathbb{R}$. Moreover, any travelling wave solution of system (3.1) connecting E^- and E^+ is a translate of φ.*

Proof. We will apply the notations in Lemma 3.3. Let $\delta_0 = \min_{z\in\mathbb{R}} \rho(z)$, and choose $\varsigma \geqslant \max\{\varsigma_0, \frac{1}{\delta_0}\}$. For $\varepsilon \in (0, \varepsilon_0(\varsigma))$, by Lemma 3.2, there exists $\tilde{t}$ such that

$$\varphi(z-\tilde{z}) - \varepsilon\delta_0 \leqslant u(z, \tilde{t}, \psi) \leqslant \varphi(z+\tilde{z}) + \varepsilon\delta_0, \quad \forall z \in \mathbb{R}.$$

Let $f(z) = u(z, \tilde{t}, \psi)$. Then, from the construction of $w^{\pm}(z,t)$ in Lemma 3.3, we have $w^-(z,0) \leqslant u(z,0,f) \leqslant w^+(z,0), \forall z \in \mathbb{R}$. By the comparison principle, we have $w^-(z,t) \leqslant u(z,t,f) \leqslant w^+(z,t), \forall z \in \mathbb{R}, t \geqslant 0$. Note that $u(z, t+\tilde{t}, \psi) = u(z, t, u(z, \tilde{t}, \psi))$. We then have

$$\varphi(z-\tilde{z}-\varepsilon\varsigma) - \varepsilon\rho(z-\tilde{z})e^{-\sigma t} \leqslant u(z, t+\tilde{t}, \psi) \leqslant \varphi(z+\tilde{z}+\varepsilon\varsigma) + \varepsilon\rho(z+\tilde{z})e^{-\sigma t}, \quad \forall t \geqslant 0. \tag{3.10}$$

Define $\Phi_t(\psi) := u(\cdot, t, \psi), \forall \psi \in \mathbb{X}_+, t \geqslant 0$. By the estimate (3.10), the positive orbit $\gamma^+(\psi) := \{\Phi_t(\psi) : t \geqslant 0\}$ is bounded in $C^1(\mathbb{R}, \mathbb{R}^2)$. Note that $\lim_{z\to\pm\infty} \varphi(z) = E^{\pm}$. Consequently, the positive orbit $\gamma^+(\psi)$ is precompact in $\mathbb{X}$, and hence its omega limit set $\omega(\psi)$ is nonempty, compact and invariant.

Letting $z_0 = \tilde{z} + \varepsilon\zeta$ and $t \to \infty$ in (3.10), we then have $\omega(\psi) \subset I := [\varphi(\cdot - z_0), \varphi(\cdot + z_0)]_{\mathbb{X}}$. Let $h(s) = \varphi(\cdot + s), \forall s \in [-z_0, z_0]$. Then h is a monotone homeomorphism from $[-z_0, z_0]$ onto a subset of I. Let $V = [E^-, E^+]_{\mathbb{X}}$. Then $\Phi_t : V \to V$ is a monotone autonomous semiflow. By Lemma 3.4, each $h(s)$ is a stable equilibrium for Φ_t. Clearly, each $\phi \in I$ satisfies condition (3.7) and hence, by the above proof, $\gamma^+(\phi)$ is procompact. By Lemma 3.5, it suffices to verify the condition 3(a) to obtain the convergence of $\gamma^+(\psi)$.

Assume that for some $s_0 \in [-z_0, z_0)$ and $\phi_0 \in I, \varphi(\cdot + s_0) <_{\mathbb{X}} \phi(\cdot)$ for all $\phi \in \omega(\phi_0)$; that is, $\varphi(\cdot + s_0) <_{\mathbb{X}} \omega(\phi_0)$. By Lemma 3.1, $\varphi(z + s_0) \ll \Phi_t(\phi)(z), \forall z \in \mathbb{R}, t > 0$, and hence, by the invariancce of $\omega(\phi_0), \varphi(z+s_0) \ll \phi(z), \forall \phi \in \omega(\phi_0), z \in \mathbb{R}$.

Since $\lim_{z\to\pm\infty} \varphi'(z) = 0$, we can choose a large positive number $z_1 \in (z_0, +\infty)$ such that $\bar{\delta} = \sup_{|z|\geqslant z_1 - z_0} \| \varphi'(z) \| \leqslant (1/4\zeta^2)$. By the compactness of $\omega(\phi_0)$, there exists $s_1 \in (s_0, z_0)$ such that $s_1 - s_0 < 2\varepsilon_0\varsigma$, and

$$\varphi(z+s_1) \ll \phi(z), \quad \forall z \in [-z_1, z_1], \quad \phi \in \omega(\phi_0).$$

For any fixed $\phi \in \omega(\phi_0)$, there exists a time sequence $\{t_j\}$ such that $\lim_{j\to\infty} t_j = +\infty$, and $\lim_{j\to\infty} \Phi_{t_j}(\phi_0) = \phi$. Fix a t_j such that $\| \Phi_{t_j}(\phi_0) - \phi \| < \bar{\delta}(s_1 - s_0)$. Since $\varphi(z+s_1) \ll \phi(z)$ for $z \in [-z_1, z_1]$, and $\varphi(z+s_0) - \varphi(z+s_1) \ll \phi(z) - \varphi(z+s_1)$ for $\forall z \in \mathbb{R}$, we have

$$\begin{aligned}
\Phi_{t_j}(\phi_0)(z) - \varphi(z+s_1) &= \Phi_{t_j}(\phi_0)(z) - \phi(z) + \phi(z) - \varphi(z+s_1) \\
&> -\bar{\delta}(s_1 - s_0)\vec{e} + \phi(z) - \varphi(z+s_1) \\
&> -\bar{\delta}(s_1 - s_0)\vec{e} - \sup_{|z|\geqslant z_1} \| \varphi(z+s_0) - \varphi(z+s_1) \| \vec{e} \\
&\geqslant -\bar{\delta}(s_1 - s_0)\vec{e} - (s_1 - s_0) \sup_{|z|\geqslant z_1} \| \varphi'(z) \| \vec{e} \\
&\geqslant -2\bar{\delta}(s_1 - s_0)\vec{e} \geqslant -\varepsilon_1 \rho(z+s_1), \quad \forall z \in \mathbb{R},
\end{aligned}$$

where $\vec{e}$ is the unit vector in $\mathbb{R}^2, \varepsilon_1 = (s_1 - s_0)/2\varsigma^2\delta_0$. Note that $\varepsilon_1 < \varepsilon_0$ and $\varepsilon_1\varsigma \leqslant (1/2)(s_1 - s_0)$. By the construction of $w^-(z,t)$ in Lemma 3.3, we have $w^-(z,0) \leqslant \Phi_{t_j}(\phi_0)(z)$. It then follows that

$$\begin{aligned}
\Phi_t(\phi_{t_j}(\phi_0))(z) &\geqslant w^-(z,t) = \varphi(z+s_1 - \varepsilon_1\varsigma(1 - e^{-\sigma t})) - \varepsilon_1\rho(z+s_1)e^{-\sigma t} \\
&\geqslant \varphi(z+s_1 - \varepsilon_1\varsigma) - \varepsilon_1\rho(z+s_1)e^{-\sigma t} \\
&\geqslant \varphi(z+s_1 - \frac{1}{2}(s_1 - s_0)) - \varepsilon_1\rho(z+s_1)e^{-\sigma t} \\
&= \varphi(z + \frac{1}{2}(s_1 + s_0)) - \varepsilon_1\rho(z+s_1)e^{-\sigma t}, \quad z \in \mathbb{R}, \ t > 0.
\end{aligned}$$

Setting $t = t_i - t_j$ and $t_i \to \infty$, we then obtain that $\varphi(\cdot + \frac{1}{2}(s_1 + s_0)) \leqslant_{\mathbb{X}} \phi(\cdot)$. Denote $s_2 = (1/2)(s_1 + s_0)$. Then $s_2 \in (s_0, s_1) \subset [s_0, z_0]$ and $\varphi(\cdot + s_2) \leqslant_{\mathbb{X}} \phi(\cdot)$. Since $\phi \in \omega(\phi_0)$ is arbitrary, we have $\phi(\cdot + s_2) \leqslant_{\mathbb{X}} \omega(\phi_0)$.

By Lemma 3.5, there exists $s_\psi \in [-z_0, z_0]$ such that $\omega(\psi) = h(s_\psi) = \varphi(\cdot + s_\psi)$. Then $\lim_{t\to\infty} \Phi_t(\psi) = \varphi(\cdot + s_\psi)$. Since $U(x, t, \psi) = u(x - ct, t, \psi) = \Phi_t(\psi)(x - ct)$, we have $\lim_{t\to\infty} \| U(x, t, \psi) - \varphi(x - ct + s_\psi) \| = 0$ uniformly for $x \in \mathbb{R}$.

Let $\tilde{\varphi}(x - \tilde{c}t)$ be a traveling wave solution of system (3.1) connecting E^- and E^+. Then $\tilde{\varphi}$ satisfies condition (3.7). By what have been proven above, there exists $\tilde{s}_\psi \in \mathbb{R}$ so that $\lim_{t\to\infty} \| \tilde{\varphi}(\cdot - \tilde{c}t) - \varphi(\cdot - ct + \tilde{s}_\psi) \| = 0$. By a change of variable $z = x - ct$, we have $\lim_{t\to\infty} \| \tilde{\varphi}(\cdot + (c - \tilde{c})t) - \varphi(\cdot + \tilde{s}_\psi) \| = 0$. Since $\tilde{\varphi}(\pm\infty) = E^\pm$ and $\varphi(\cdot)$ is strictly increasing on $\mathbb{R}$, we must have $\tilde{c} = c$, and hence, $\tilde{\varphi}(\cdot) = \varphi(\cdot + \tilde{s}_\psi)$. □

4. GLOBAL EXPONENTIAL STABILITY

In Section 3 we proved that for a large class of initial values, solutions of (3.1) converge to translates of the travelling wave front. In this section, we will show that this convergence is also uniformly exponential via the spectrum analysis.

A standard technique for determining stability (exponential) of travelling waves is to use the linearization criterion. As in Section 3, we assume that system (3.1) admits a strictly increasing travelling wave solution $U(x, t) = \varphi(x - ct) = (\varphi_1(x - ct), \varphi_2(x - ct)), c \neq 0$. If the right-hand side of (3.6) is linearized about its equilibrium solution $\varphi(z)$, the resulting linear operator is

$$(Lu)(z) = \begin{pmatrix} \mathrm{d}u_{1,zz} + cu_{1,z} \\ cu_{2,z} \end{pmatrix} + J_\varphi(z) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

where $J_\varphi(z) = (F^i_j(\varphi(z))), u(z) = (u_1(z), u_2(z)) \in \mathbb{X}$.

The linearization criterion for stability of the travelling wave front is that the spectrum $\sigma(L)$ of L (except for zero) lies in a left-half complex plane and is bounded away from the imaginary axis, and zero is a simple eigenvalue. Note that zero is always an eigenvalue of L because of the translation invariance of travelling waves. For the point spectrum $\sigma_p(L)$ of L, we have the following result.

Lemma 4.1. *Assume that λ is an eigenvalue of L with eigenfunction $u \in \mathbb{X}_c$, complexified $\mathbb{X}$. If $u \notin span\{\varphi'(\cdot)\}$, then $Re\lambda < 0$.*

Proof. Let $D = diag(d, 0), C = diag(c, c), B(z) = (F^i_j(\varphi(z)))$ and $B^\pm = (F^i_j(E^\pm))$.We claim that there exist positive vectors $q^\pm$ such that $B^\pm q^\pm < 0$.

Note that the reaction system (3.2) is cooperate and $E^{\pm}$ are stable nodes. Since B^{+} is irreducible, we can choose q^{+} as a positive eigenvector associated with the negative principle eigenvalue of B^{+}. Thus, $B^{+}q^{+} < 0$. If $g'(0) > 0$, then B^{-} is an irreducible matrix. Therefore, a positive eigenvector q^{-} can be chosen such that $B^{-}q^{-} < 0$. If $g'(0) = 0$, let $q^{-} = (1, \varepsilon)$. Then $B^{-}q^{-} < 0$ for some sufficiently small positive number ε.

Let $z_0 > 0$ be a sufficiently large number so that $B(z)q^{+} < 0$ for $z \geqslant z_0$, and $B(z)q^{-} < 0$ for $z \leqslant -z_0$. Set $\epsilon > 0$ be small so that $(\epsilon^2 D + \epsilon C + B(z))q^{+} < 0$ for $z \geqslant z_0$, and $(\epsilon^2 D + \epsilon C + B(z))q^{-} < 0$ for $z \leqslant -z_0$. Letting $Q^{\pm}(z) = e^{\pm\epsilon z}q^{\pm}$, we have $LQ^{+} < 0$ for $z \geqslant z_0$, and $LQ^{-} < 0$ for $z \leqslant -z_0$.

Assume that λ is an eigenvalue of L with eigenfunction $u \in \mathbb{X}_c$ and $u \notin span\{\varphi'(\cdot)\}$. Rewrite $\lambda = \lambda_1 + \lambda_2 i, u = u^1 + u^2 i$, where $\lambda_1, \lambda_2 \in \mathbb{R}, u^1, u^2 \in \mathbb{X}$, and $u^2 = 0$ if $\lambda_2 = 0$. Consider the Cauchy problem:

$$v_t(z,t) = Lv(z,t) - \lambda_1 v(z,t), \qquad v(z,0) = u^1(z).$$

The function $v(z,t) = u^1(z)\cos(\lambda_2 t) - u^2(z)\sin(\lambda_2 t)$ is a solution of this problem. We require that at least one of the elements of the vecator-valued function $v(z,t)$ takes a positive value (otherwise, we can consider $-v(z,t)$). Let $\psi(z) = \varphi'(z) > 0$. Since $v(z,t)$ is periodic and bounded, we can choose a positive number r such that

$$v_t(z,t) \leqslant r\psi(z) \quad \text{for } |z_1| \leqslant z_0 \quad \text{and } t \geqslant 0, \tag{4.1}$$

where for at least one $k = 1$ or 2, and one $|z_i| \leqslant z_0$ and $t_1 > 0$, we have the following equality for the kth components

$$v_k(z_1, t_1) = r\psi_k(z_1). \tag{4.2}$$

We proceed the proof by contradication. Suppose that $\lambda_1 \geqslant 0$. Then the following two claims hold.

Claim 1: $v(z,t) \leqslant r\psi(z)$ for all $z \in \mathbb{R}, t \geqslant 0$. Suppose, by contradication, that there exist some $z > z_0, t \geqslant 0$ such that $v(z,t) > r\psi(z)$. Since $Q^{+}(z) = e^{ez}q^{+} \to +\infty$ as $z \to +\infty$, there exists $\tilde{r} > 0$ such that $v(z,t) \leqslant r\psi(z) + \tilde{r}Q^{+}(z)$ for $z \geqslant z_0, t \geqslant 0$, where at least for one j, one $z_2 > z_0$ and $t_2 > 0$, we have the equality for jth component:

$$v_j(z_2, t_2) = r\psi(z_2) + \tilde{r}Q_j^{+}(z_2).$$

Let $y(z,t) = r\psi(z) + \tilde{r}Q^{+}(z) - v(z,t)$. Then the jth component $y_j(z,t)$ satisfies $y_j(z_2, t_2) = 0, y_j(z_0, t) > 0, y_j(z,t) \geqslant 0$ for $z \geqslant z_0, t \geqslant 0$. Therefore,

$y_{j,t}(z_2,t_2) \leqslant 0, y_{j,z}(z_2,t_2) = 0$, and if $j=1$, then $y_{j,zz}(z_2,t_2) \geqslant 0$. Since $L\psi(z)=0$, and $LQ^+(z)<0$ for $z \geqslant z_0$, $y_j(z,t)$ satisfies

$$\begin{aligned} y_{j,t} &= -v_{j,t} = -(Lv - \lambda_1 v)_j \\ &> (-Lv + \lambda_1 v + Lr\psi + L\tilde{r}Q^+ - \lambda_1(r\psi + \tilde{r}Q^+))_j \\ &= (Ly - \lambda_1 y)_j \\ &= d_j y_{j,zz} + c y_{j,z} + F_1^j(\varphi(z)) y_1 + F_2^j(\varphi(z)) y_2 - \lambda_1 y_j, \end{aligned}$$

where $d_j = d$ if $j=1$ and $d_j=0$ if $j=2$. Evaluating the above inequality at (z_2,t_2) and using the positivity of $F_i^j(\varphi(z))$ for $i \neq j$, we then have a contradiction in signs. Thus $v(z,t) \leqslant r\psi(z)$, $\forall z \geqslant z_0, t \geqslant 0$. Using the same argument, we obtain that $v(z,t) \leqslant r\psi(z)$, $\forall z \leqslant z_0, t \geqslant 0$. Thus the claim is established.

Claim 2: $v(z,t) \equiv r\psi(z)$, $\forall z \in \mathbb{R}, t \geqslant 0$. Suppose, by contradiction, that $v(z,t) \not\equiv r\psi(z)$. Then there exist $\bar{t} > 0$, $\bar{z} \in \mathbb{R}$ and $\bar{k}$ such that $v_{\bar{k}}(\bar{z},\bar{t}) < r\psi(\bar{z})$. Let $Y(z,t) = r\psi(z) - v(z,t)$. Then $Y(z,t) > 0$ for $z \in \mathbb{R}, t \geqslant 0$, and $Y_{\bar{k}}(\bar{z},\bar{t}) > 0$. Moreover, the components $Y_i(z,t)$ of $Y(z,t)$ satisfy

$$\begin{aligned} Y_{i,t} &\geqslant (LY - \lambda_1 Y)_i \\ &= d_i Y_{i,zz} + cY_{i,z} + F_1^i(\varphi(z))Y_1 + F_2^i(\varphi(z))Y_2 - \lambda_1 Y_i, \end{aligned} \tag{4.3}$$

By a similar argument as in Claim 1, it follows from the inequality (4.3) that $Y_i(\bar{z},\bar{t}) > 0$ for each $i=1,2$. Applying the strict positivity theorem [33, Theorem 5.5.4], we have $Y_1(z,t) > 0$ for $z \in \mathbb{R}, t > \bar{t}$. By the periodicity of Y in t, we have $Y_1(z,t) > 0$ for $z \in \mathbb{R}, t \geqslant 0$. Therefore, if $k=1$, defined by (4.2), we then have a contradiction. Let us consider the case where $k=2$. Since $Y_2(z_1,t_1)=0$ and $Y_2(z,t) \geqslant 0$ for $z \in \mathbb{R}, t \geqslant 0$, it follows that $Y_{2,t}(z_1,t_1) \leqslant 0, Y_{2,z}(z_1,t_1)=0$. Note that $Y_1(z_1,t_1) > 0$. Evaluating (4.3) with $i=2$ at (z_1,t_1), we have a contradiction in signs. This established the claim.

For $\lambda_2 \neq 0$, Claim 2 implies that $Lv(z,t) = Lr\psi(z) = 0$, i.e., $Lu^1(z)\cos(\lambda_2 t) - Lu^2(z)\sin(\lambda_2 t) = 0$, $\forall t \geqslant 0$. Hence $Lu^1(z)=0$ and $Lu^2(z)=0$. Therefore, $Lu=0$, which contradicts the fact that $Lu = \lambda u \neq 0$. For $\lambda_2 = 0$, we have $u_2 = 0$ and hence $u(z) = u^1(z) = v(z,t) = r\psi(z)$, which contradicts our assumption that $u \notin span\{\psi\}$. Therefore, $\lambda_1 =$ Re $\lambda < 0$. □

To show that the essential spectrum $\sigma_e(L)$ of L satisfies the linearization criterion, we will use the results developed in [21, pp. 136–138].

Let T be the following linear operator:

$$(Tu)(z) = \begin{pmatrix} \mathrm{d}u_{1,zz} + cu_{1,z} \\ cu_{2,z} \end{pmatrix} + J(z)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

where $J(z) = (F^i_j(E^+))$ for $z \geqslant 0$, $j(z) = (F^i_j(E^-))$ for $z < 0$, and $u(z) = (u_1(z), u_2(z)) \in \mathbb{X}$. Consider the eigenvalue problem of T

$$(T-\lambda I)\begin{pmatrix} p \\ r \end{pmatrix} = 0, \tag{4.4}$$

where

$$\begin{pmatrix} p \\ r \end{pmatrix}(z) \in \mathbb{X}_c,$$

complexified $\mathbb{X}$. Rewrite (4.4) as a system

$$\begin{aligned} p'(z) &= q, \\ q'(z) &= -\frac{1}{d}(cq + J_{11}(z)p + J_{12}(z)r - \lambda p), \\ r'(z) &= -\frac{1}{c}(J_{21}(z)p + J_{22}(z)r - \lambda r), \end{aligned}$$

where $J(z) = (J_{ij}(z))$. Let $y = (p, q, r) \in \mathbb{C}^3$, and write the above system as

$$y' = A(z, \lambda)y, \tag{4.5}$$

where

$$A(z,\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{d}(J_{11}(z)-\lambda) & -\frac{c}{d} & -\frac{1}{d}J_{12}(z) \\ -\frac{1}{d}J_{21}(z) & 0 & -\frac{1}{c}(J_{22}(z)-\lambda) \end{pmatrix}.$$

Define $A^+(\lambda) := A(1,\lambda)$, $A^-(\lambda) := A(-1,\lambda)$, and $S^\pm = \{\lambda \in \mathbb{C} : A^\pm = A^\pm(\lambda)$ have imaginary eigenvalues$\}$, which will provide the necessary information about $\sigma_e(L)$.

Lemma 4.2. *$\mathbb{C}\backslash S^\pm$ has an open connected set G for which there exists a $\lambda_0 < 0$ such that $\{\lambda : \operatorname{Re}\lambda > \lambda_0\} \subset G$.*

Proof. Let $P(\lambda) = det(A^\pm - \mu i I)$. Then

$$\begin{aligned} P(\lambda) &= \det\begin{pmatrix} -\mu i & 1 & 0 \\ -\frac{1}{d}\left(F^1_1(E^\pm)-\lambda\right) & -\frac{c}{d}-\mu i & -\frac{1}{d}F^1_2(E^\pm) \\ -\frac{1}{d}F^2_1(E^\pm) & 0 & -\frac{1}{c}\left(F^2_2(E^\pm)-\lambda\right)-\mu i \end{pmatrix} \\ &= -\frac{1}{cd}\lambda^2 + \left(\frac{2}{d}\mu i - \frac{1}{c}\mu^2 + \frac{1}{cd}\left(F^1_1(E^\pm)+F^2_2(E^\pm)\right)\right)\lambda \\ &\quad +\mu^2\left(\mu i + \frac{c}{d}\right) + \frac{1}{c}\mu^2 F^2_2(E^\pm) - \frac{1}{d}\mu i\left(F^1_1(E^\pm)+F^2_2(E^\pm)\right) \\ &\quad +\frac{1}{cd}\left(F^2_1(E^\pm)F^1_2(E^\pm) - F^1_1(E^\pm)F^2_2(E^\pm)\right). \end{aligned}$$

Setting $P(\lambda)=0$, we have

$$\lambda=\frac{1}{2}\left(F_1^1(E^{\pm})+F_2^2(E^{\pm})+2c\mu i-\mathrm{d}\mu^2\right)\pm\sqrt{\Delta},$$

where $\Delta=\left(F_2^2(E^{\pm})-F_1^1(E^{\pm})+d\mu^2\right)^2+4(F_1^2(E^{\pm})F_2^1(E^{\pm})$, which is positive since $F_j^i(u_1,u_2)\geqslant 0$ for $i\neq j, 1\leqslant i, j\leqslant 2$. Let $\lambda=\lambda_1+\lambda_2 i$, where $\lambda_1,\lambda_2\in\mathbb{R}$. Then

$$\lambda_1=\frac{1}{2}\left(F_1^1(E^{\pm})+F_2^2(E^{\pm})-\mathrm{d}\mu^2\right)\pm\sqrt{\Delta},\ \lambda_2=c\mu.$$

Eliminating the parameter μ, we have

$$\lambda_1=\frac{1}{2}\left(F_1^1(E^{\pm})+F_2^2(E^{\pm})-\frac{d}{c^2}\lambda_2^2\right)\\ \pm\frac{1}{2}\sqrt{\left(F_2^2(E^{\pm})-F_1^1(E^{\pm})+\frac{d}{c^2}\lambda_2^2\right)^2+4F_1^2(E^{\pm})F_2^1(E^{\pm})}.$$

Thus the set $S^{\pm}$ is symmetric about the real axis in the complex plane. It is easy to obtain that the derivative $\mathrm{d}\lambda_1/\mathrm{d}\lambda_2\leqslant 0$ for $\lambda_2\geqslant 0$. Therefore, the maximal real part of the point in $S^{\pm}$ is one of the following values

$$\lambda^{\pm}=\frac{1}{2}\left(F_1^1(E^{\pm})+F_2^2(E^{\pm})\right)+\frac{1}{2}\sqrt{\left(F_2^2(E^{\pm})-F_1^1(E^{\pm})\right)^2+4F_1^2(E^{\pm})F_2^1(E^{\pm})}.$$

Note that $\lambda^{\pm}$ are exact eigenvalues of the Jacobian matrix of the reaction system (3.2) at $E^{\pm}$. Thus $\lambda^{\pm}<0$. Therefore, the curves $S^{\pm}$ are bounded uniformly away from the imaginary axis. This proves the lemma. □

The implication of Lemma 4.2 is that there is no essential spectrum point of L in G.

Lemma 4.3. $\sigma(L)\cap G\subset\sigma_p(L)$.

Proof. Define the differential operator $L(\lambda)y=y'-A(z,\lambda)y$. Then, by [21, Lemma 2, p. 138], one of the following cases holds: (i) $0\in\sigma(L(\lambda))$ for all $\lambda\in G$ (defined in Lemma 4.2); (ii) $0\in\rho(L(\lambda))$ for all $\lambda\in G$ except for isolated points, and the exception points are poles of $L(\lambda)^{-1}$ of finite order. Therefore, the set G consists either entirely of spectral points $\sigma(T)$ of T (case (i)), or entirely of normal points of T (case (ii)). Here a normal point is a resolvent point or an isolated eigenvalue of T with finite multiplicity. It is not difficult to see that large positive numbers are not eigenvalues of T (see, e.g., the proof of Lemma 4.1). Thus, G consists entirely of normal points of T. Let $S=J_\varphi(z)-J(z)$. Then $L=T+S$. It is easy to show that $S(\lambda_0 I-T)^{-1}$ is compact for large positive λ_0. By [21, Theorem A.1, p. 136], G consists either entirely of normal points of L, or entirely of eigenvalues of L. Hence, Lemma 4.1 implies that $\sigma(L)\cap G\subset\sigma_p(L)$. □

Now we know that $\sigma_e(L)$ causes no problem for linear stability. Hence, we can draw the following conclusion about the global exponential stability.

Theorem 4.1. *Let $\varphi(x-ct)$ be a monotone traveling wave solution of (3.1) with $c\neq 0$. Then there exists a positive constant $\mu>0$ such that for every $\psi\in\mathbb{X}_+$ satisfying (3.7), the solution $U(x,t,\psi)$ of (3.1) satisfies*

$$\| U(x,t,\psi)-\varphi(x-ct+s_\psi)\|\leqslant C_\psi e^{-\mu t},\quad \forall x\in\mathbb{R},\ t\geqslant 0,$$

for some constant $s_\psi\in\mathbb{R}$ and $C_\psi>0$.

Proof. By Lemmas 4.1 and 4.3, it follows that zero is a simple eigenvalue of L and the rest of the spectrum $\sigma(L)$ lies in the left-hand complex plane and is bounded away from the imaginary axis. Thus, by the main theorem in [16], zero solution is stable for the linearized PDE system of (3.1) at the travelling wave solution. Then by the result in [14], the travelling wave solutions are locally exponentially stable for the original system (3.1), and hence, Theorem 3.1 completes the proof. □

5. NUMERICAL SIMULATIONS

By Theorems 2.1, 3.1 and 4.1, we see that the epidemic model (2.1) admits a unique monotone bistable travelling wave solution (up to translation), which is globally exponentially stable with phase shift. In order to check this result, we numerically simulate solutions of system (2.1). Assume that $d=0.2, \alpha=2.3, \beta=1$ and $g(z)=z^2/(1+z^2)$. Then, $a=0.5821, b=1.7179$, and the integral (2.12) is $0.07521>0$. Hence, Theorem 2.1 implies that the wave speed c^* is positive. System (2.1) is discretised by using the finite difference method on a finite spatial interval $[-L, L]$ with the Neumann boundary condition, where $L > 0$ is sufficiently large in comparison with the domain in which the solutions rapidly change shapes. The numerical wave profile is shown as solid lines in Fig. 3 and 4. Figures 5 and 6 provide the evolution of the solution with initial function being the dashed lines in Fig. 3 and 4. We can see that the solution rapidly converges to the wave profile as the time t becomes large.

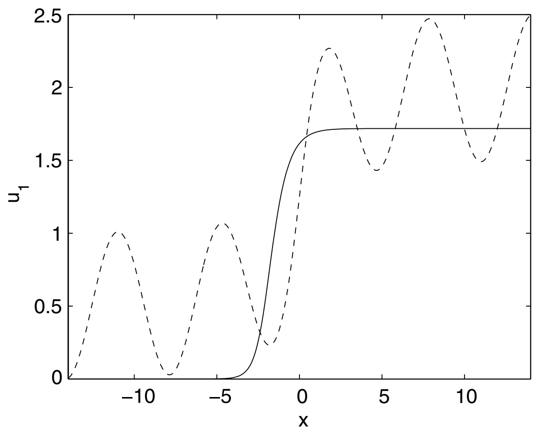

Figure 3. u_1 components.

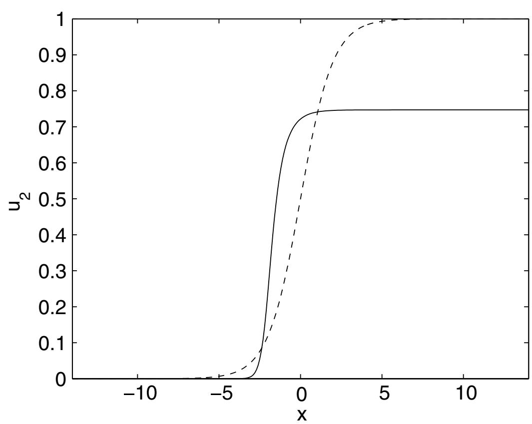

Figure 4. u_2 components.

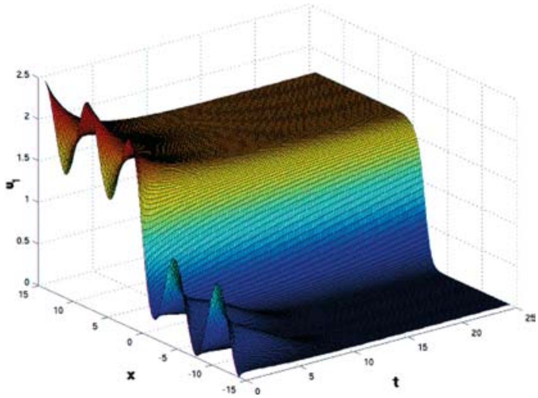

Figure 5. The graph of $u_1(x, t)$.

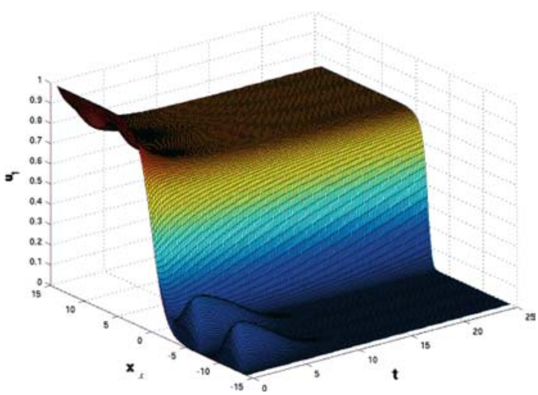

Figure 6. The graph of $u_2(x, t)$.

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