Generating Orthogonal Polynomials and their Derivatives using Vertex|Matching-Partitions of Graphs

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Abstract

A vertex|matching-partition $(V|M)$ of a simple graph G is a spanning collection of vertices and independent edges of G. Let vertex $v \in V$ have weight w_v and edge $e \in M$ have weight w_e. Then the weight of $V|M$ is $w(V|M) = \prod_{v \in V} w_v \cdot \prod_{e \in M} w_e$. Define the vertex|matching-partition function of G as $\mathcal{W}(G) = \sum_{V|M} w(V|M)$. In this paper we study this function when G is a path and a cycle. We generate all orthogonal polynomials as vertex|matching-partition functions of suitably labelled paths, and indicate how to find their derivatives in some cases. Here Taylor's Expansion is used and an application to associated polynomials is given. We also give a combinatorial interpretation of coefficients in the case of multiplicative and additive weights. Results are extended to the weighted cycle.

1 Vertex|Matching-Partition Function, 3-Term Recurrence for Orthogonal Polynomials

In this paper we define a vertex|matching-partition and the vertex|matching-partition function of a simple graph. The vertex|matching-partition function of a path was briefly discussed without development in Viennot [7, pp. VI-9], (see also Viennot [8]). Here we develop this idea further by showing how to generate any orthogonal polynomial as the vertex|matching-partition function of a suitably weighted path; many examples are given. We extend our results to the vertex|matching-partition function of a weighted cycle. Finally, we show how to generate derivatives of some orthogonal polynomials, and consider an application to the associated polynomials of Van Assche [6].

Let G be a simple graph with vertices $V(G)$ and edges $E(G)$. A *matching* M of G is a set of edges in $E(G)$, no two of which are incident. For a matching M of G, let $V_M = V$ denote the vertices of G which are not incident to any edge in M. Call the pair $V|M$ a *vertex|matching-partition* or a *vm-partition* of G. For every $v \in V(G)$ and every vm-partition $V|M$ of G, either $v \in V$ or v is incident to an edge in M which (with a slight abuse of notation) we write as $v \in M$, but not both. Call V the *vertex-set* and M the *matching* of the vm-partition $V|M$. Let $V_\emptyset$ denote the empty vertex-set, and $M_\emptyset$ the empty matching.

Let $v \in V(G)$ have *weight* w_v, and let $e = (u,v) \in E(G)$ have weight $w_e = w_{(u,v)}$. Now define the *weight* of a vm-partition $V|M$ as

$$w(V|M) = \prod_{v\in V} w_v \cdot \prod_{e \in M} w_e, \tag{1}$$

where products over $V_\emptyset$ and $M_\emptyset$ are 1.

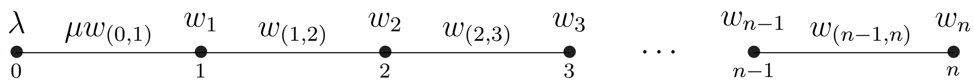
Finally define the *vm-partition function* of G as:

$$\mathcal{W}(G) = \sum_{V|M} w(V|M),$$

where the summation is over all vm-partitions $V|M$ of G.

Weighted Path $P(n)$

Let λ and μ be arbitrary variables, and let $P(n)$ be the weighted path on the $n+1$ vertices $\{0, 1, \ldots, n\}$, see below. The vertex labels $\{0, 1, \ldots, n\}$ are shown below the path. The weights of the vertices and edges are shown above the path. We call this Model I.



Let $\mathcal{P}_n$ be the *vm*-partition function of $P(n)$, *i.e.*,

$$\mathcal{P}_n = \mathcal{W}(P(n)).$$

With $\mathcal{P}_{-1} = \mu$ and $\mathcal{P}_0 = \lambda$ we have,

Theorem 1.1 *For* $n \geq 1$,

$$\mathcal{P}_n = w_n \mathcal{P}_{n-1} + w_{(n-1,n)} \mathcal{P}_{n-2}. \tag{2}$$

Proof. Note that $\mathcal{P}_0 = \mathcal{W}(P(0)) = \lambda$ and $\mathcal{P}_1 = \mathcal{W}(P(1)) = \lambda w_1 + \mu w_{(0,1)}$. Now, for $n \geq 1$, consider vertex n, the last vertex of $P(n)$; it has weight w_n. Let $V|M$ be an arbitrary *vm*-partition of $P(n)$, so either $n \in V$ or $n \in M$.

If $n \in V$ then w_n occurs as a factor of $w(V|M)$. Upon factoring out w_n we obtain the weight of a *vm*-partition of $P(n-1)$; conversely, given the weight of any *vm*-partition of $P(n-1)$, its product with w_n gives the weight of a *vm*-partition of $P(n)$ which contains n in its vertex-set. This accounts for the first term in the right-hand side of (2).

If $n \in M$ then $w_{(n-1,n)}$ occurs as a factor in $w(V|M)$ and, by a similar argument to above, the sum of the weights of all such *vm*-partitions is $w_{(n-1,n)}\mathcal{P}_{n-2}$, the second term in the right-hand side of (2). ∎

Thus

$$\begin{aligned}
\mathcal{P}_n &= w_n \mathcal{P}_{n-1} + w_{(n-1,n)} \mathcal{P}_{n-2} \\
&= \sum_{\substack{V|M \\ n \in V}} w(V|M) + \sum_{\substack{V|M \\ n \in M}} w(V|M),
\end{aligned}$$

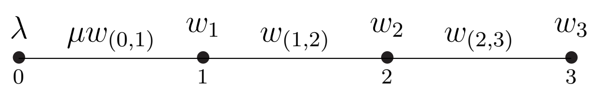
where the first summation is over all vm-partitions of $P(n)$ which contain vertex n in its vertex-set, V; the second over all vm-partitions of $P(n)$ which contain n in its matching, M. We say that equation (2) comes from *decomposing* $\mathcal{P}_n$ at vertex n.

Thus any sequence of polynomials which obey the 3-term recurrence (2) can be obtained as the sequence of vm-partition functions of $P(n)$; in particular orthogonal polynomials obey such a 3-term recurrence.

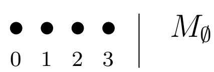
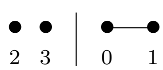
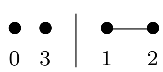
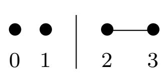
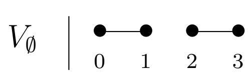
Example 1

$\mathcal{P}_0 = \mathcal{W}(P(0)) = \lambda$
$\mathcal{P}_1 = \mathcal{W}(P(1)) = \lambda w_1 + \mu w_{(0,1)}$
$\mathcal{P}_2 = \mathcal{W}(P(2)) = \lambda w_1 w_2 + \mu w_2 w_{(0,1)} + \lambda w_{(1,2)}$
$\mathcal{P}_3 = \mathcal{W}(P(3)) = \lambda w_1 w_2 w_3 + \mu w_2 w_3 w_{(0,1)} + \lambda w_3 w_{(1,2)} + \lambda w_1 w_{(2,3)} + \mu w_{(0,1)} w_{(2,3)}$.

For example $\mathcal{P}_3$ comes from $P(3)$, shown below

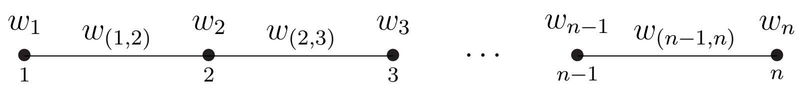


which has 5 vm-partitions:

vm-partition	weight
	$\lambda w_1 w_2 w_3$
	$w_2 w_3 \mu w_{(0,1)}$
	$\lambda w_3 w_{(1,2)}$
	$\lambda w_1 w_{(2,3)}$
	$\mu w_{(0,1)} w_{(2,3)}$

We now consider the two fundamental solutions to recurrence (2), f_n and g_n.

The first fundamental solution, f_n, of recurrence (2) is the solution with initial values $f_{-1} = \mu = 0$ and $f_0 = \lambda = 1$. Thus, from Model I, decomposing $\mathcal{P}_n$ at vertex 0, f_n is the vm-partition function of



(3)

The second fundamental solution, g_n, has $g_{-1} = \mu = 1$ and $g_0 = \lambda = 0$. Thus, using Model I again, $g_n = w_{(0,1)} \times$ the vm-partition function of the path

(4)

Now, decomposing $\mathcal{P}_n$ at vertex 0, gives:

Theorem 1.2 *For $n \geq 1$,*

$$\mathcal{P}_n = \lambda f_n + \mu g_n.$$

■

Remark 1.3 Let G be a graph with $V(G) = \{1, \ldots, n\}$ with vertex weights w_v for $v \in V(G)$, and edge weights w_e for any edge $e \in E(G)$, and with vm-partition function $\mathcal{G}$.
(i) Suppose we replace any particular vertex weight $w_{v'}$ with $\eta w_{v'}$, and replace the weights $w_{e'}$ on all edges e' incident to v' with weight $\eta w_{e'}$, then the vm-partition function of this new graph is $\eta\mathcal{G}$.
(ii) Extending (i), suppose we replace every vertex weight w_v with ηw_v, and replace every edge weight w_e with weight $\eta^2 w_e$, then the vm-partition function of this graph is $\eta^n \mathcal{G}$.

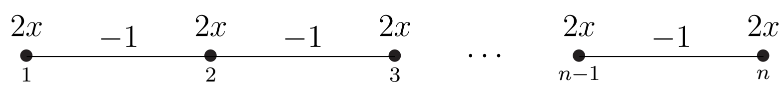
As mentioned above, orthogonal polynomials obey a 3-term recurrence, the next few examples involve well-known orthogonal polynomials.

Example 2 Chebyshev polynomials. $U_{-1}(x) = 0$ and $U_0(x) = 1$.

Consider the recurrence below for $n \geq 1$,

$$\mathcal{P}_n = 2x\mathcal{P}_{n-1} - \mathcal{P}_{n-2}. \tag{5}$$

The Chebyshev polynomials of the second kind, $U_n(x)$, are the first fundamental solution of this recurrence, *i.e.*, $U_n(x) = f_n(x)$, $(\mu = U_{-1} = 0, \lambda = U_0 = 1)$. Thus, comparing (5) with (2), then using (3) with $w_n = 2x$ for $n \geq 1$ and $w_{(n-1,n)} = -1$ for $n \geq 2$, we see that $U_n(x)$ is the vm-partition function of

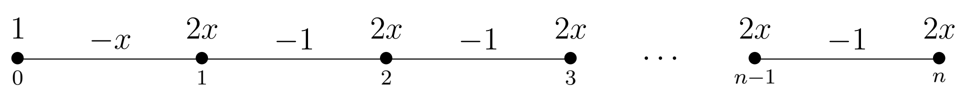


(6)

The number of k-matchings in $P(n)$ is $\binom{n-k}{k}$, see Godsil [3, p. 2], so

$$U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k}. \quad \text{(See Riordan [4, p. 59].)}$$

The Chebyshev polynomials of the first kind, $T_n(x)$, are given by $T_n(x) = f_n(x) + x\, g_n(x)$, $(\mu = T_{-1} = x, \lambda = T_0 = 1)$. Thus $T_n(x)$ is the vm-partition function of



(7)

Now decompose $T_n(x)$ at vertex 0 which has weight 1. This gives $T_n(x) = 1 \cdot U_n(x) + (-x) \cdot U_{n-1}(x)$, *i.e.*,

$$T_n(x) = U_n(x) - x\, U_{n-1}(x),$$

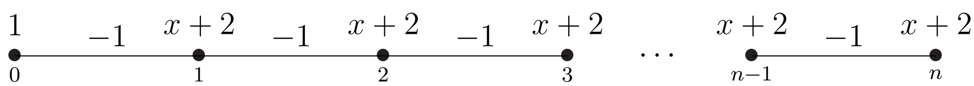
a well-known formula which involves both types of Chebyshev polynomials. We derive the explicit formula for $T_n(x)$ later in Example 7.

Example 3 Associated Legendre polynomials. $\rho_{-1}(x) = 1$ and $\rho_0(x) = 1$.

Consider the recurrence below for $n \geq 1$,

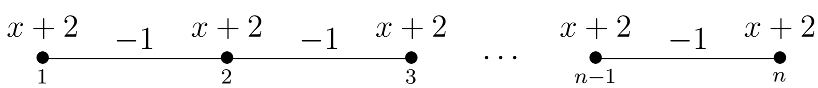
$$\mathcal{P}_n = (x+2)\mathcal{P}_{n-1} - \mathcal{P}_{n-2}.$$

With $\mu = \rho_{-1} = 1$ and $\lambda = \rho_0 = 1$ we get the associated Legendre polynomials, $\rho_n(x) = f_n(x) + g_n(x)$, see Riordan [4, p. 66] . Thus, $\rho_n(x)$ is the vm-partition function of



(8)

$U_n(\frac{x}{2}+1)$ is the vm-partition function of



Decomposing the vm-partition function of (8), $\rho_n(x)$, at vertex 0 yields

$$\rho_n(x) = U_n\left(\frac{x}{2}+1\right) - U_{n-1}\left(\frac{x}{2}+1\right),$$

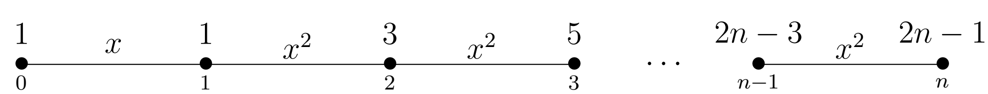
for $n \geq 0$, see Riordan [4, p. 85] .

Example 4 Bessel polynomials. $\theta_{-1}(x) = \frac{1}{x}$ and $\theta_0(x) = 1$.

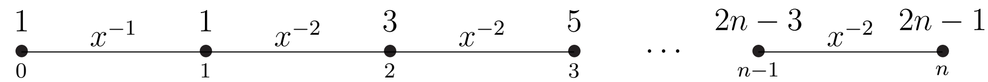
Consider the recurrence below for $n \geq 1$,

$$\mathcal{P}_n = (2n-1)\mathcal{P}_{n-1} + x^2\mathcal{P}_{n-2}.$$

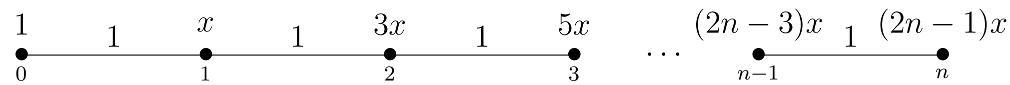
The Bessel polynomials, $\theta_n(x)$, are the solutions of this recurrence with $\mu = \theta_{-1} = \frac{1}{x}$ and $\lambda = \theta_0 = 1$, *i.e.*, $\theta_n(x) = f_n(x) + \frac{1}{x}g_n(x)$. Thus, $\theta_n(x)$ is the vm-partition function of



For the polynomials $\theta_n(x^{-1})$ the model is



Let $y_n(x) = x^n\theta_n(x^{-1})$ be the *reciprocal* Bessel polynomials (see Riordan [4, p. 77]), then, modifying Remark 1.3(i) for the n vertices $\{1, \ldots, n\}$ with $\eta = x$, we see that $y_n(x)$ is the vm-partition function of



By decomposing at vertex n we can derive the recurrence for $y_n = y_n(x)$,

$$y_n = (2n-1)xy_{n-1} + y_{n-2},$$

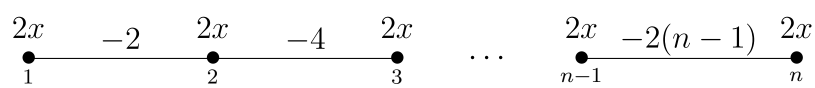
with $y_{-1} = y_0 = 1$.

Example 5 Hermite polynomials. $H_{-1}(x) = 0$ and $H_0(x) = 1$.

Consider the recurrence below for $n \geq 1$,

$$\mathcal{P}_n = 2x\mathcal{P}_{n-1} - (2n-2)\mathcal{P}_{n-2}.$$

The Hermite polynomials, $H_n(x)$, are the first fundamental solution of this recurrence, *i.e.*, $H_n(x) = f_n(x)$, $(\mu = H_{-1} = 0, \lambda = H_0 = 1)$. Thus, using (3), $H_n(x)$ is the vm-partition function of



Now the coefficient of x^{n-2k} in the vm-partition function of the above weighted path is

$$2^{n-2k}(-2)^k \sum_{N_k} \prod_{i \in N_k} i \tag{9}$$

where N_k is a k-subset of $\{1, \ldots, n-1\}$ in which no two elements are consecutive. Using (9) and the well-known formula:

$$H_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{n!}{k!\,(n-2k)!}\,(2x)^{n-2k},$$

we have:

Corollary 1.4 *Let N_k be a k-subset of $\{1, \ldots, n-1\}$ in which no two elements are consecutive. Then*

$$\sum_{N_k} \prod_{i \in N_k} i \;=\; \frac{n!}{k!\,(n-2k)!\,2^k}.$$

$\blacksquare$

For example, if $n = 7$, $k = 3$, then N_3 can be any of the four subsets $\{1, 3, 5\}$, $\{1, 3, 6\}$, $\{1, 4, 6\}$, or $\{2, 4, 6\}$. We have

$$1 \cdot 3 \cdot 5 + 1 \cdot 3 \cdot 6 + 1 \cdot 4 \cdot 6 + 2 \cdot 4 \cdot 6 = \frac{7!}{3!\,1!\,2^3} 5.$$

Remark 1.5 Note the polynomials He_n which satisfy the recurrence $He_n = x\,He_{n-1} - (n-1)\,He_{n-2}$. This recurrence arises by dividing all of the above weights by 2, and the relation $H_n(x) = 2^{n/2}\,He_n(x\sqrt{2})$ corresponds to Remark 1.3(ii).

Example 6 q-Lommel polynomials. $U_{-1}(x; a, b) = 0$ and $U_0(x; a, b) = 1$.

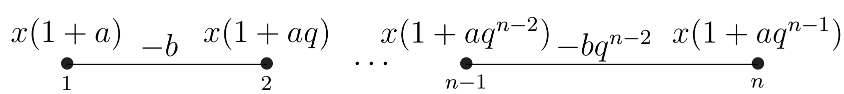
Consider the recurrence below for $n \geq 1$,

$$\mathcal{P}_n = x\,(1 + aq^{n-1})\mathcal{P}_{n-1} - bq^{n-2}\mathcal{P}_{n-2}.$$

Then the first fundamental solution of this recurrence, $f_n(x)$, are the polynomials $U_n(x; a, b)$ which appear in Al-Salam and Ismail [1].

For example $U_1(x; a, b) = x\,(1+a)$ and $U_2(x; a, b) = x^2\,(1+a)(1+aq) - b$.

Using (3), $U_n(x; a, b)$ is the vm-partition function of



Note that, as stated in Al-Salam and Ismail [1], $U_n(\frac{2}{x}; -q^v, q^v)$ are the q-Lommel polynomials of Ismail [5].

Weighted Cycle $C(n)$

The weighted path Model I generates $\mathcal{P}_n = \lambda f_n + \mu g_n$ where f_n is the first fundamental solution of recurrence (2) and g_n is the second fundamental solution. Consider $C(n)$, the cycle with n vertices and n edges, weighted as shown in Fig. 1. Call this Model II.

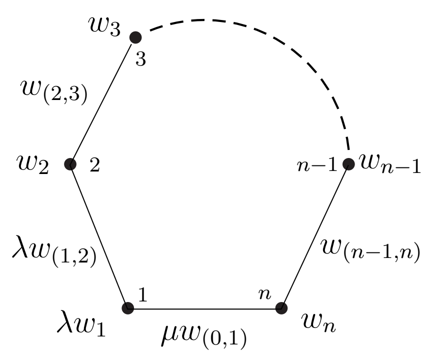


Figure 1. Weighted $C(n)$

Let $\mathcal{C}_n$ be the vm-partition function of $C(n)$, *i.e.*,

$$\mathcal{C}_n = \mathcal{W}(C(n)).$$

Compare Theorem 1.2.

Theorem 1.6 *For* $n \geq 1$*,*

$$\mathcal{C}_n = \lambda f_n + \mu g_{n-1}.$$

Proof. Let us decompose $\mathcal{C}_n$ at vertex n. This yields 3 types of vm-partitions of $C(n)$. (a) Those for which vertex n lies in the vertex set. From Model II above, and using (3) for f_{n-1} and Remark 1.3 (i) on vertex 1 with $\eta = \lambda$, we see that these vm-partitions sum to $w_n \cdot \lambda f_{n-1}$. (b) Those which contain edge $(n-1, n)$; these sum to $w_{(n-1,n)} \cdot \lambda f_{n-2}$. Thus the vm-partitions in (a) and (b) together sum to $\lambda(w_n f_{n-1} + w_{(n-1,n)} f_{n-2}) = \lambda f_n$. (c) Those which contain edge $(n, 1)$. The sum of these is μg_{n-1}, using (4).

The vm-partitions in (a), (b), and (c) partition the set of vm-partitions of $C(n)$, thus $\mathcal{C}_n = \lambda f_n + \mu g_{n-1}$, as required. ∎

Example 7 Here we use Model II to generate $T_n(x)$, the Chebyshev polynomials of the first kind, and derive the explicit form for $T_n(x)$ as mentioned in Example 2.

Let $T_n = \lambda f_n + \mu g_{n-1}$, where f_n and g_n are the fundamental solutions of (5). The values $T_0 = 1$ and $T_1 = x$ give $\lambda = \mu = \frac{1}{2}$ *i.e.*, $T_n = \frac{1}{2} f_n + \frac{1}{2} g_{n-1}$. Thus $T_n(x)$ is the vm-partition function of the weighted cycle in Fig. 2.

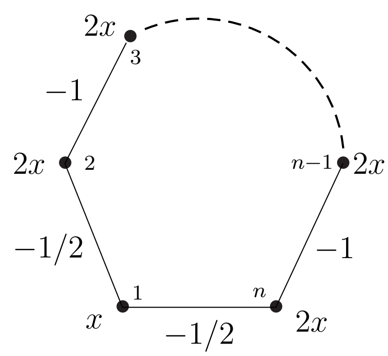


Figure 2. Chebyshev weights

And, using Remark 1.3(i) at vertex 1 with $\eta = 2$, we see that $2\,T_n(x)$ is the vm-partition function of the weighted cycle in Fig. 3.

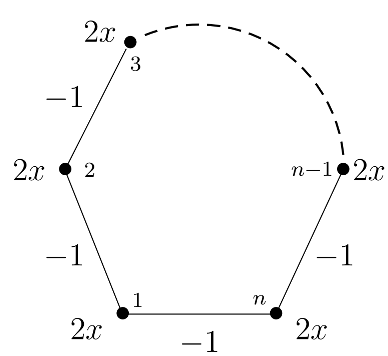


Figure 3. Chebyshev weights

See Godsil [3, p. 144].

The number of k-matchings in $C(n)$ is $\frac{n}{n-k}\binom{n-k}{k}$, see Godsil [3, p. 14], so

$$2\,T_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (2x)^{n-2k}.$$

Now decompose $2\,T_n(x)$ at *edge* $(n,1)$. A vm-partition of Fig. 3 which does not contain edge $(n,1)$ is a vm-partition of the path (6), the sum of all such vm-partitions is $U_n(x)$. Similarly, the sum of the vm-partitions of Fig. 3 which contain edge $(n,1)$ is $-U_{n-2}(x)$. Thus we get

$$2T_n(x) = U_n(x) - U_{n-2}(x),$$

the well-known relation between the two types of Chebyshev polynomials, see Riordan [4, p. 59] .

Example 8 Let t and u be arbitrary variables and consider the recurrence

$$\mathcal{P}_n = (t+u)\mathcal{P}_{n-1} - tu\mathcal{P}_{n-2}. \tag{10}$$

Now $A_n = t^n + u^n$ is a solution with $A_{-1} = t^{-1} + u^{-1} = \frac{t+u}{tu}$ and $A_0 = 2$. Thus $A_n = 2f_n + \frac{t+u}{tu} g_n$ where f_n and g_n are the fundamental solutions to (10), and we could use Model I to generate A_n. However, a better generation is given by Model II: solving $A_n = \lambda f_n + \mu g_{n-1}$ at $n = 0$ yields $\lambda = \mu = 1$. Thus, for $n \geq 1$, $A_n = f_n + g_{n-1}$ is the vm-partition function of the weighted cycle in Fig. 4.

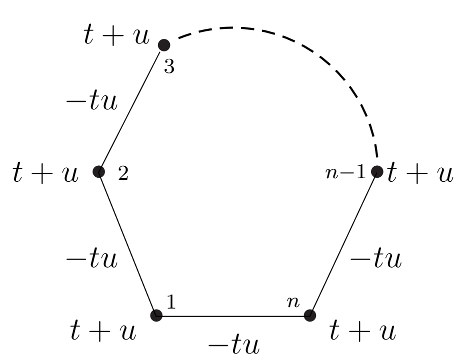


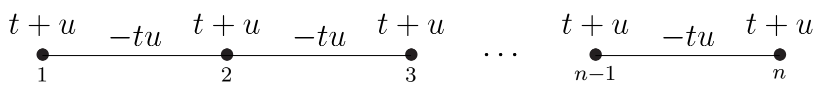
Figure 4. General Chebyshev weights

Similar to Example 7 we have,

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (t+u)^{n-2k} (tu)^k = t^n + u^n,$$

a 'very old' identity, see Riordan [4, p. 58] .

It seems worthwhile to mention that f_n is the vm-partition function of



and it is straightforward to prove by induction that $f_n = \sum_{i=0}^{n} t^{n-i}\, u^i = \dfrac{t^{n+1} - u^{n+1}}{t-u}$, which is valid for $t \neq u$. Now the matchings polynomial of P_n is $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} x^k$, thus we have,

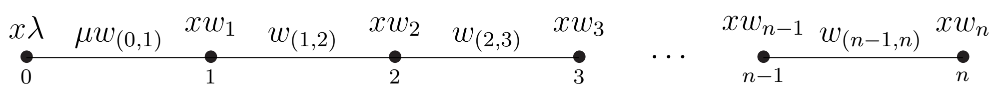
$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} (t+u)^{n-2k} (tu)^k = \sum_{i=0}^{n} t^{n-i}\, u^i = \frac{t^{n+1} - u^{n+1}}{t-u},$$

which is valid for $t \neq u$.

2 Derivatives of vm-Partition Functions

Vertex Weights containing x

Consider Model I in which each vertex weight has been multiplied by a variable x. The other weight parameters λ, w_v, and $w_{(u,v)}$ do not contain x. This gives the weighted path $\overline{P}(n)$ as shown below. Call this Model $\bar{\text{I}}$, the case of multiplicative weights.



Let $\overline{\mathcal{P}}_n(x) = \overline{\mathcal{P}}_n = \mathcal{W}(\overline{P}(n))$ be the vm-partition function of this weighted path, and let $\overline{\mathcal{P}}_n^{(k)}$ denote its k-th derivative with respect to x for $k \geq 1$.

Let $V_k = \{v_1, \ldots, v_k\}$ be a set of k distinct vertices of $\overline{P}(n)$ and let $\overline{P}(n) - V_k$ be the graph obtained when these k vertices and their incident edges are removed. The empty graph is the graph with no vertices and no edges, let its vm-partition function be 1.

Before the main result in this section we need:

For each $i = 1, \ldots, m$ let S_i be a weighted path. Let $S = S_1 \cup S_2 \cup \cdots \cup S_m$ be a disjoint union of m such paths. Let S^* denote the path obtained from S by joining the last vertex of S_1 to the first vertex of S_2, then the last vertex of S_2 to the first vertex of S_3, ... , and so on. Let these $m-1$ new edges all have weight 0. Then, it is straightforward to prove,

Lemma 2.1

$$\mathcal{W}(S) = \mathcal{W}(S^*).$$

■

Now the main result in this section. (See Godsil [3, p. 2] for a similar result with $k = 1$.)

Theorem 2.2 *For any $n \geq 0$ and $k \geq 1$, the k-th derivative of $\overline{\mathcal{P}}_n$ is given by*

$$\overline{\mathcal{P}}_n^{(k)} = k! \sum_{V_k} w_{v_1} \cdots w_{v_k} \mathcal{W}(\overline{P}(n) - V_k), \tag{11}$$

where the sum is over every $V_k = \{v_1, \ldots, v_k\}$, a k-set of vertices of $\overline{P}(n)$.

Proof. For $k = 1$ denote $\overline{\mathcal{P}}_n^{(1)}$ by $\overline{\mathcal{P}}'_n$, we use induction on n. First, for $n = 0$, we have $\overline{\mathcal{P}}_0 = x\lambda$ and so $\overline{\mathcal{P}}'_0 = \lambda$ as the left-hand side of (11); and the right-hand side equals $\lambda\mathcal{W}(\overline{P}(0) - \{0\}) = \lambda$ also, because $\overline{P}(0) - \{0\}$ is the empty graph. For $n = 1$ we have $\overline{\mathcal{P}}_1 = x^2\lambda w_1 + \mu w_{(0,1)}$, and $\overline{\mathcal{P}}'_1 = 2x\lambda w_1$ as the left-hand side of (11); the right-hand side equals $\lambda\mathcal{W}(\overline{P}(1) - \{0\}) + w_1\mathcal{W}(\overline{P}(1) - \{1\}) = \lambda x w_1 + w_1 x\lambda = 2x\lambda w_1$, as required.

For $n \geq 2$, $\overline{\mathcal{P}}_n = xw_n\overline{\mathcal{P}}_{n-1} + w_{(n-1,n)}\overline{\mathcal{P}}_{n-2}$. So, using the product rule for derivatives at the first line, the induction hypothesis at the second line, and a modification of (2) to include the vm-partition function of a union of 2 paths (see Lemma 2.1 with $m = 2$) at the fourth line, we have

$$\begin{aligned}
\overline{\mathcal{P}}'_n &= w_n\overline{\mathcal{P}}_{n-1} + xw_n\overline{\mathcal{P}}'_{n-1} + w_{(n-1,n)}\overline{\mathcal{P}}'_{n-2}, \\
&= w_n\overline{\mathcal{P}}_{n-1} + xw_n\sum_{v=0}^{n-1} w_v\mathcal{W}(\overline{P}(n-1) - \{v\}) \\
&\qquad + w_{(n-1,n)}\sum_{v=0}^{n-2} w_v\mathcal{W}(\overline{P}(n-2) - \{v\}) \\
&= w_n\overline{\mathcal{P}}_{n-1} + xw_nw_{n-1}\mathcal{W}(\overline{P}(n-1) - \{n-1\}) \\
&\qquad + \sum_{v=0}^{n-2} w_v\{xw_n\mathcal{W}(\overline{P}(n-1) - \{v\}) + w_{(n-1,n)}\mathcal{W}(\overline{P}(n-2) - \{v\})\} \\
&= w_n\overline{\mathcal{P}}_{n-1} + xw_nw_{n-1}\overline{\mathcal{P}}_{n-2} + \sum_{v=0}^{n-2} w_v\mathcal{W}(\overline{P}(n) - \{v\}) \\
&= \sum_{v=0}^{n} w_v\mathcal{W}(\overline{P}(n) - \{v\}),
\end{aligned}$$

as required. So (11) is true for $k = 1$.

Now we induct on k, so assume (11) is true for k, then

$$\begin{aligned}
\overline{\mathcal{P}}_n^{(k)} &= k! \sum_{V_k} w_{v_1} \cdots w_{v_k} \mathcal{W}(\overline{P}(n) - V_k), \quad \text{and} \\
\overline{\mathcal{P}}_n^{(k+1)} = \overline{\mathcal{P}}_n^{(k)\prime} &= k! \sum_{V_k} w_{v_1} \cdots w_{v_k} \mathcal{W}(\overline{P}(n) - V_k)'.
\end{aligned}$$

Now the graph $\overline{P}(n) - V_k$ is a disjoint union of paths, hence, from Lemma 2.1 and the above,

$$\overline{\mathcal{P}}_n^{(k+1)} = k! \sum_{V_k} w_{v_1} \cdots w_{v_k} \left(\sum_{v_{k+1} \notin V_k} w_{v_{k+1}} \mathcal{W}((\overline{P}(n) - V_k) - \{v_{k+1}\}) \right).$$

Each set $V_k \cup \{v_{k+1}\}$ will appear $k+1$ times, so

$$\overline{\mathcal{P}}_n^{(k+1)} = (k+1)! \sum_{V_{k+1}} w_{v_1} \cdots w_{v_{k+1}} \mathcal{W}(\overline{P}(n) - V_{k+1}),$$

where the sum is over every $V_{k+1} = \{v_1, \ldots, v_{k+1}\}$, a $(k+1)$-set of vertices of $\overline{P}(n)$; thus the induction goes through and (11) is true for all $k \geq 1$. ■

From Taylor's Expansion we have the following Corollaries, in which the notation $x = y$ means replace x by y, etc:

Corollary 2.3 *The polynomials $\overline{\mathcal{P}}_n(x)$ satisfy the identity*

$$\overline{\mathcal{P}}_n(x+y) = \sum_{k=0}^{n+1} x^k \sum_{V_k} w_{v_1} \cdots w_{v_k} \mathcal{W}(\overline{P}(n) - V_k) \Big|_{x=y}$$

■

Setting $y = 0$ gives

Corollary 2.4 *The vm-partition function for $\overline{P}(n)$ with multiplicative weights is*

$$\overline{\mathcal{P}}_n(x) = \sum_{k=0}^{n+1} x^k \sum_{V_k} w_{v_1} \cdots w_{v_k} \mathcal{W}(\overline{P}(n) - V_k) \Big|_{x=0}$$

■

Thus, we have a combinatorial interpretation of the coefficients of x^k in the case of multiplicative weights.

A special case of Corollary 2.4 is given by

$$\mathcal{P}_n = \overline{\mathcal{P}}_n(1) = \sum_{k=0}^{n+1} \sum_{V_k} w_{v_1} \cdots w_{v_k} \mathcal{W}(\overline{P}(n) - V_k) \Big|_{x=0}$$

which corresponds to arranging the terms of $\mathcal{P}_n$ so that we sum over the vm-partitions $V|M$ where $|V| = k$ for $k = 0, \ldots n+1$.

We now give an alternative derivation of the above formula in Corollary 2.4.

A *perfect matching* of a graph with an even number of vertices is a set of edges, no two of which are incident and which cover every vertex exactly once. Clearly a path with an even number of vertices has a unique perfect matching obtained by choosing its left-most edge and then every second edge.

We say that the graph $\overline{P}(n) - V_k$ for any $k = 0, \ldots, n+1$ is composed of $k+1$ *segments*. Each segment is a path; we allow for an empty path if two of the vertices in V_k are adjacent in $\overline{P}(n)$, or if vertex $0 \in V_k$ then the first segment in $\overline{P}(n) - V_k$ is the empty path, similarly if vertex $n \in V_k$ then the last segment in $\overline{P}(n) - V_k$ is the empty path.

Now clearly $\mathcal{W}(\overline{P}(n) - V_k)|_{x=0} = 0$ unless each of the $k+1$ segments in $\overline{P}(n) - V_k$ is either the empty path or a path with an even number of vertices, since in a path with an odd number of vertices *every* vm-partition must contain at least one isolated vertex and hence every vm-partition has weight 0. In this former case we have $\mathcal{W}(\overline{P}(n) - V_k)|_{x=0} = \prod_{e \in M} w_e$, where M is the following set of edges: the first edge in the first non-empty segment of $\overline{P}(n) - V_k$ followed by every second edge in this segment, then the first edge in the second non-empty segment of $\overline{P}(n) - V_k$ followed by every second edge in this segment,..., and so on. That is, M is the unique perfect matching of $\overline{P}(n) - V_k$.

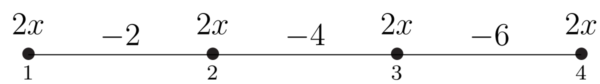
Now, from (1), modified for the case of multiplicative weights, a typical term in $\overline{\mathcal{P}}_n(x)$ is $\prod_{v \in V} x^{|V|} w_v \cdot \prod_{e \in M} w_e$, where we now see that M is the unique perfect matching of $\overline{P}(n) - V$. Hence, setting $|V| = k$ we have:

$$\begin{aligned}
\overline{\mathcal{P}}_n(x) &= \sum_{k=0}^{n+1} \sum_{V_k} \prod_{v \in V_k} x^k w_v \cdot \prod_{e \in M} w_e \\
&= \sum_{k=0}^{n+1} x^k \sum_{V_k} w_{v_1} \cdots w_{v_k} \mathcal{W}(\overline{P}(n) - V_k) \Big|_{x=0}
\end{aligned}$$

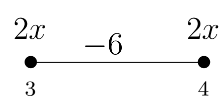
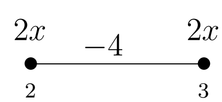
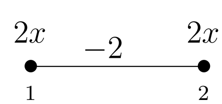
as above.

The polynomials $U_n(x)$ from Example 2, $H_n(x)$ from Example 5, and $U_n(x; a, b)$ from Example 6 can be generated by Model $\overline{\mathrm{I}}$.

Example 9 $H_4(x)$ is the vm-partition function of

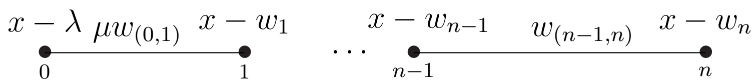


So $H_4(x) = 16x^4 - 48x^2 + 12$ and its second derivative $H_4^{(2)}(x) = 192x^2 - 96$.

$V_2 = \{v_1, v_2\}$	$\overline{P}(4) - V_2$	$w_{v_1} w_{v_2} \mathcal{W}(\overline{P}(4) - V_2)$
$\{1, 2\}$		$2 \cdot 2 \cdot (4x^2 - 6)$
$\{1, 3\}$		$2 \cdot 2 \cdot 4x^2$
$\{1, 4\}$		$2 \cdot 2 \cdot (4x^2 - 4)$
$\{2, 3\}$		$2 \cdot 2 \cdot 4x^2$
$\{2, 4\}$		$2 \cdot 2 \cdot 4x^2$
$\{3, 4\}$		$2 \cdot 2 \cdot (4x^2 - 2)$
		$96x^2 - 48$

But $k = 2$ so $2! \cdot (96x^2 - 48) = 192x^2 - 96 = H_4^{(2)}(x)$.

We have a similar result for Model I with vertex weights $x-\lambda$ on vertex 0 and $x-w_v$ on vertex $v \in \{1,2,\ldots,n\}$, where λ, w_v, and $w_{(u,v)}$ do not contain x. Call this weighted path $\widehat{P}(n)$ and let $\widehat{\mathcal{P}}_n$ be its vm-partition function; it is shown below. We call this Model $\widehat{\mathrm{I}}$, the case of additive weights.

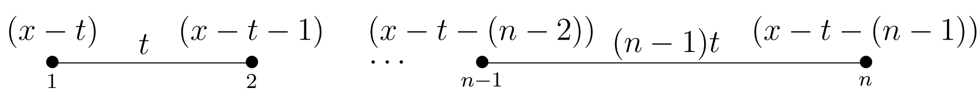


Example 10 Poisson-Charlier polynomials. $P_{-1}(x)=0$ and $P_0(x)=1$.

Consider the recurrence below for $n \geq 1$,

$$\mathcal{P}_n = (x-t-n+1)\mathcal{P}_{n-1} + t(n-1)\mathcal{P}_{n-2}.$$

Then the first fundamental solution of this recurrence are the Poisson-Charlier polynomials $P_n(x,t)$, which can be generated by Model $\widehat{\mathrm{I}}$, shown below.



We have the following Theorem, the proof of which is similar to that of Theorem 2.2.

Theorem 2.5 *For any $n \geq 0$ and $k \geq 1$, the k-th derivative of $\widehat{\mathcal{P}}_n$ is given by*

$$\widehat{\mathcal{P}}_n^{(k)} = k! \sum_{V_k} \mathcal{W}(\widehat{P}(n) - V_k),$$

where the sum is over every V_k, a k-set of vertices of $\widehat{P}(n)$. ■

and as for Theorem 2.2 we have the Corollaries:

Corollary 2.6 *The polynomials $\widehat{\mathcal{P}}_n(x)$ satisfy the identity*

$$\widehat{\mathcal{P}}_n(x+y) = \sum_{k=0}^{n+1} x^k \sum_{V_k} \mathcal{W}(\widehat{P}(n) - V_k)\bigg|_{x=y}$$

■

and

Corollary 2.7 *The vm-partition functionfor $\widehat{P}(n)$ with additive weights is*

$$\widehat{\mathcal{P}}_n(x) = \sum_{k=0}^{n+1} x^k \sum_{V_k} \mathcal{W}(\widehat{P}(n) - V_k) \bigg|_{x=0}$$

∎

As before, this gives a combinatorial interpretation of the coefficients of x^k in the case of additive weights.

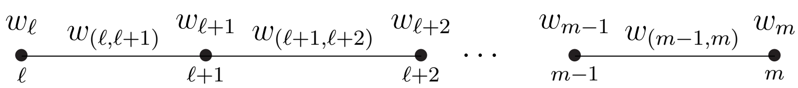
We observe that:

$$\mathcal{W}(\widehat{P}(n) - V_k) \bigg|_{x=0} = \mathcal{W}(\overline{P}(n) - V_k) \bigg|_{x=-1}$$

Finally, see Feinsilver, McSorley, and Schott [2] for an application of Theorem 2.5 to Lommel polynomials.

Segment Polynomials

Consider Model I. For any $0 \leq \ell \leq m \leq n$ let $P(\ell, m)$ denote the weighted subpath of $P(n)$ starting at vertex ℓ and ending at vertex m,



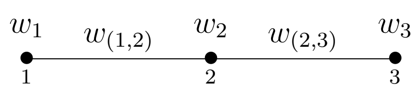
Now define

$$\mathcal{P}_{\ell,m} = \mathcal{W}(P(\ell, m))$$

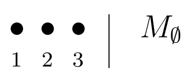
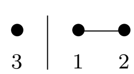
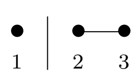
to be the vm-partition function of $P(\ell, m)$.

Example 11

Starting with $P(3)$ we have the weighted subpath $P(1, 3)$:



which has 3 vm-partitions:

vm-partition	weight
	$w_1 w_2 w_3$
	$w_3 w_{(1,2)}$
	$w_1 w_{(2,3)}$

Thus $\mathcal{P}_{1,3} = w_1 w_2 w_3 + w_3 w_{(1,2)} + w_1 w_{(2,3)}$.

Note that $\mathcal{P}_{0,n} = \mathcal{P}_n$ for $n \geq 0$. We also define $\mathcal{P}_{0,-1} = \mathcal{P}_{-1} = \mu$. Then, with the starting conditions $\mathcal{P}_{\ell,\ell-2} = 0$ and $\mathcal{P}_{\ell,\ell-1} = 1$, we have for $m \geq 1$,

$$\mathcal{P}_{\ell,m} = w_m \mathcal{P}_{\ell,m-1} + w_{(m-1,m)} \mathcal{P}_{\ell,m-2}.$$

Example 12 With natural notation we see, from the weighted path (6) in Example 2, that $U_{\ell,m}(x) = U_{m-\ell+1}(x)$, for $\ell \geq 1$, these are Chebyshev segment polynomials of the second kind. Similarly, from the weighted path (7) we have $T_{\ell,m}(x) = U_{m-\ell+1}(x)$, for $\ell \geq 1$, the Chebyshev segment polynomials of the first kind. We also have $\rho_{\ell,m}(x) = \rho_{m-\ell+1}(x+2)$, for $\ell \geq 1$, the associated Legendre segment polynomials.

For orthogonal polynomials satisfying the three-term recurrence

$$x\, p_n(x) = a_{n+1}\, p_{n+1}(x) + b_n\, p_n(x) + a_n\, p_{n-1}(x)$$

with $p_{-1}(x) = 0$, $p_0(x) = 1$, i.e., the first fundamental solution to the recurrence, the ℓ^{th} *associated polynomials*, $p_{(n,\ell)}(x)$ are the first fundamental solution to the recurrence

$$x\, p_{(n,\ell)}(x) = a_{n+\ell+1}\, p_{(n+1,\ell)}(x) + b_{n+\ell}\, p_{(n,\ell)}(x) + a_{n+\ell}\, p_{(n-1,\ell)}(x)$$

(see Van Assche [6], with our notation a slight variant). In other words, the subscripts on the coefficients are shifted up ℓ units. This corresponds exactly to moving up ℓ vertices along the path, namely our segment polynomials. Thus we have a (new) combinatorial interpretation of the associated polynomials.

Equation 2.10 of Theorem 1 of [6] gives the relation

$$p_n'(x) = \sum_{j=1}^{n} \frac{1}{a_j}\, p_{j-1}(x)\, p_{(n-j,j)}(x). \tag{12}$$

This is the case $k = 1$ of our Theorem 2.2 which expands the first derivative of $\overline{\mathcal{P}}_n(x)$ in terms of segment polynomials. Explicitly, we have

$$\overline{\mathcal{P}}_n' = \sum_{v=0}^{n} w_v \mathcal{P}_{0,v-1} \mathcal{P}_{v+1,n} = \sum_{v=0}^{n} w_v \mathcal{P}_{v-1} \mathcal{P}_{v+1,n}. \tag{13}$$

Furthermore, our model gives formulas for 2^{nd} and higher derivatives of $\overline{\mathcal{P}}_n$ in terms of more complicated combinations of segment polynomials.

Example 13 Note that equation (12) suggests the form of a convolution. An example that illustrates this is given by the Gegenbauer polynomials, $C_n^\lambda(x)$, with generating function

$$G(x, v; \lambda) = (1 - 2xv + v^2)^{-\lambda} = \sum_{n=0}^{\infty} v^n C_n^\lambda(x).$$

Differentiating both sides with respect to x gives

$$2v\lambda\, G(x, v; \lambda + 1) = \sum_{n=0}^{\infty} v^n (C_n^\lambda(x))'.$$

Or, for $n \geq 1$,

$$(C_n^\lambda(x))' = 2\lambda\, C_{n-1}^{\lambda+1}(x).$$

Observe that the relation $G(x, v; \lambda + \mu) = G(x, v; \lambda)G(x, v; \mu)$ may be read as the statement that the sequence $\{C_n^{\lambda+\mu}\}_{n\geq 0}$ is the convolution of the sequences $\{C_n^\lambda\}_{n\geq 0}$ and $\{C_n^\mu\}_{n\geq 0}$. Now, the case $\mu = 1$ gives $C_n^\lambda = U_n$, Chebyshev polynomials of the second kind. Thus,

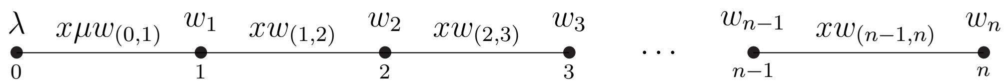
$$(C_n^\lambda(x))' = 2\lambda\, C_{n-1}^{\lambda+1}(x) = 2\lambda \sum_{j=1}^{n} C_{n-j}^\lambda(x)\, U_{j-1}(x)$$

holds in general. In particular, taking $\lambda = 1$, recalling equation (12) to the effect that the U_n are their own associated/segment polynomials, we have an exact correspondence with equations (12) and (13) for Chebyshev polynomials of the second kind.

Edge Weights containing x

Finally we consider derivatives with respect to variables on the *edges* of Model I.

Consider Model I in which each weight has been multiplied by the variable x. As before, other weight parameters do not contain x. Let $\overline{Q}(n)$ denote this weighted path and let $\overline{\mathcal{Q}}_n$ be its *vm*-partition function, see below.



Let $M_k = \{e_1, \dots, e_k\}$ be a k-matching of $\overline{Q}(n)$, then we have the following theorem, the proof of which is similar to the above proof of Theorem 2.2.

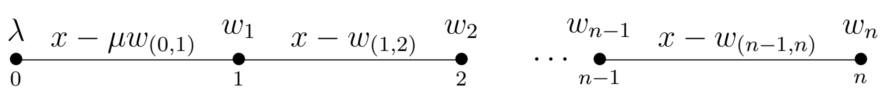
Theorem 2.8 *For any $n \geq 0$ and $k \geq 1$, the k-th derivative of $\overline{\mathcal{Q}}_n$ is given by*

$$\overline{\mathcal{Q}}_n^{(k)} = k! \sum_{M_k} w_{e_1} \cdots w_{e_k} \mathcal{W}(\overline{Q}(n) - M_k),$$

the sum is over every $M_k = \{e_1, \dots, e_k\}$, a k-matching of $\overline{Q}(n)$. ∎

And Taylor's Expansion gives results analogous to Corollaries 2.3 and 2.4.

Finally, proceeding as before, call the weighted path with edge weights as shown below $\widehat{Q}(n)$, and call its *vm*-partition function $\widehat{\mathcal{Q}}_n$.



Theorem 2.9 *For any $n \geq 0$ and $k \geq 1$, the k-th derivative of $\widehat{\mathcal{Q}}_n$ is given by*

$$\widehat{\mathcal{Q}}_n^{(k)} = k! \sum_{M_k} \mathcal{W}(\widehat{Q}(n) - M_k),$$

where the sum is over every M_k, a k-matching of $\widehat{Q}(n)$. ∎

Similarly there are results analogous to Corollaries 2.6 and 2.7.

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