There Are No Non-Trivially Uniformly (t, r)-Regular Graphs for $t > 2$.

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Abstract

A finite simple graph is uniformly (t, r)-regular if it has at least t vertices and the open neighbor set of each set of t of its vertices is of cardinality r. If $t > 1$, such a graph is trivially uniformly (t, r)-regular if either it is a matching $(t = r)$ or r is the number of non-isolated vertices in the graph. We prove the result stated in the title.

1 Uniform (t, r)-regularity

All graphs will be finite and simple, in this paper, and notation will largely be as in [10]. If G and H are graphs, $V(G)$ is the vertex set of G, $G+H$ is the disjoint union of G and H, and for a positive integer m, $mG = G+\cdots+G$ (m summands). If $u \in V(G)$, $N_G(u) = \{v \in V(G) | u \text{ and } v \text{ are adjacent in } G\}$, and if $S \subseteq V(G)$, $N_G(S) = \bigcup_{u \in S} N_G(u)$, the open neighbor set of S in G. The order of G will be denoted by $n(G)(= |V(G)|)$, or just n, if G is the only graph in the discussion.

G is uniformly (t, r)-regular if $1 \leq t \leq n$ and for each $S \subseteq V(G)$ with $|S| = t$, $|N_G(S)| = r$. This property of graphs was introduced in [4] as "(t, r)-regularity"; the problem with that terminology is that it is also used for a seemingly similar but rather less exigent property, introduced in [3] and written on in [2], [5], and [7]. In [6] the word "strong" plays the role we assign to "uniform" here; we abandon that terminology because it misleadingly suggests an analogy with strong regularity of graphs. There is a powerful connection between the two when $t = 2$ (see [9]), but the analogy at the definitional level is distant.

Uniform $(1, r)$-regularity is just plain r-regularity. When $t > 1$ there are two easily found classes of uniformly (t, r)-regular graphs:

(i) $G = mK_2$ for some $m \geq t/2$, a matching. In this case, $t = r$.

(ii) $r = n(G_1)$, where G_1 is the subgraph of G induced by the non-isolated vertices of G, and t is "sufficiently large". Indeed, as noted in [6], if $r = n(G_1) > 0$ and $n(G) - \delta(G_1) + 1 \leq t \leq n(G)$ then G is uniformly (t, r)-regular, but G is not uniformly $(n(G) - \delta(G_1), r)$-regular. And if $r = n(G_1) = 0$ then $G = nK_1$ and is uniformly $(t, 0)$-regular for all $t = 1, \ldots, n$.

For $t > 1$, uniform (t, r)-regularity due to either condition (i) or (ii) will be called *trivial*, and the big question (raised in [6]) is: are there non-trivially uniformly (t, r)-regular graphs, and, if so, what are they?

This question has been satisfactorily answered for $t = 2$. Any "strongly regular graph with $\lambda = \mu > 0$", that is, a regular graph G, say with degree $d > 0$, not complete, for which there exists μ such that for any two distinct u, $v \in V(G)$, $|N_G(u) \cap N_G(v)| = \mu$, is non-trivially uniformly $(2, 2d - \mu)$-regular. There are infinitely many such graphs (see, e.g., [8]), and it has recently been shown [9] that there are no other non-trivially uniformly $(2, r)$-regular graphs besides these. Here we settle the question for $t > 2$. The proof of the following theorem is postponed until section 3.

Theorem 1 *If $t > 2$ then for no r does there exist a non-trivially uniformly (t, r)-regular graph.*

2 An excursion into designs

If $n \geq t > 0$, an (n, t, λ)-design is a pair $(V, \mathcal{B})$ where V is a set with n elements ("points") and $\mathcal{B} = [B(i)|i \in I]$ is an indexed collection of subsets of V ("blocks") such that for each $T \subseteq V$ with $|T| = t$, $|\{i \in I | T \subseteq B(i)\}| = \lambda$. (That is, any t points of V lie together in exactly λ blocks.) We require $\mathcal{B}$ to be an indexed collection because we want to allow "repeated blocks"; that is, it may be that $B(i) = B(j)$ even though $i \neq j$. Also note that there is no requirement that the blocks be of the same size. Given such a design, let $b = |I|$, the number of blocks.

Fisher's Inequality [1, Theorem 2.6, p.66] *If $(V, \mathcal{B})$ is an $(n, 2, \lambda)$-design with $\lambda > 0$ and V not appearing as a block, then $b \geq n$.*

Theorem 2 *If $t > 2$, $\lambda > 0$, and $(V, \mathcal{B})$ is an (n, t, λ)-design with V not appearing as a block, then $b \geq n$ with equality if and only if $\mathcal{B}$ can be re-indexed to be $[V \backslash \{v\} | v \in V]$.*

Proof. We go by induction on t, starting with $t = 3$. For each $v \in V$, let $I(v) = \{i \in I | v \in B(i)\}$ and consider the derived design $(V \backslash \{v\}, \mathcal{B}'(v))$, where $\mathcal{B}'(v) = [B(i) \backslash \{v\} | i \in I(v)]$. Each derived design is an $(n-1, 2, \lambda)$-design (because $t = 3$) and $V \backslash \{v\}$ does not appear in $\mathcal{B}'(v)$ because V does not appear in $\mathcal{B}$. By Fisher's inequality, $b'(v) = |I(v)| \geq n - 1$. On the other hand, $b'(v) \leq b$.

If $b = n - 1$, then $b'(v) = n - 1 = b$, for every $v \in V$, so $I(v) = I$ for every v. But then $v \in B(i)$ for every $i \in I$, and every v, so, not only does V appear in $\mathcal{B}$, it is equal to $B(i)$ for each i, wildly contrary to hypothesis. So $b \geq n$, as asserted. Suppose that $b = n$. Then $b'(v) = |I(v)| = n$ or $n - 1$ for each $v \in V$–i.e., v is in every block of $\mathcal{B}$ or in every block but one.

On the other hand, each block of $\mathcal{B}$ is missing some element of V. Think of a bipartite graph with bipartition V, I, with $v \in V$ adjacent to $i \in I$ if and only if $v \notin B(i)$. Then each $v \in V$ has degree ≤ 1 in this graph, and each $i \in I$ has degree ≥ 1, and $|V| = n = b = |I|$. Thus the bipartite graph is a matching, and $\mathcal{B}$, possibly after renaming, is $[V \backslash \{v\} | v \in V]$.

Now suppose that $t > 3$. With $I(v)$ and $\mathcal{B}'(v)$, $v \in V$, defined as above, each derived design $(V \backslash \{v\}, \mathcal{B}'(v))$ is an $(n-1, t-1, \lambda)$-design, with $V \backslash \{v\}$ not among the blocks in $\mathcal{B}'(v)$. By the induction hypothesis, $b \geq b'(v) = |\mathcal{B}'(v)| \geq n - 1$ for each $v \in V$. From here the proof proceeds as in the case $t = 3$.

3 Proof of Theorem 1

Lemma 1 *If $t > 1$ and G is non-trivially uniformly (t, r)-regular, then G has no isolated vertices.*

Proof. Suppose that u is an isolated vertex of G. Let G_1 be the subgraph of G induced by the non-isolated vertices of G. Since G is non-trivial, $0 < r < n(G_1)$, and, therefore, $t < n(G_1)$. Let S be a $(t-1)$-subset of $V(G_1)$, and $T = S \cup \{u\}$; then $|N_G(T)| = |N_G(S)| = r$. Since $r < n(G_1)$, there is some $w \in V(G_1) \backslash N_G(S)$, and, by the definition of G_1, some $v \in V(G_1)$ adjacent to w. But then $|S \cup \{v\}| = t$ while $|N_G(S \cup \{r\})| \geq r + 1$, contradicting the assumption that G is uniformly (t, r)-regular.

The main idea that starts the proof of Theorem 1 is due to Khodkar and Leach [8]. Suppose that G is non-trivially (t,r)-regular, $t \geq 3$. For $v \in V(G)$, let $B(v) = V(G) \backslash N_G(v)$, and $\mathcal{B} = [B(v) | v \in V(G)]$. By the Lemma, no $v \in V(G)$ is isolated, so $B(v) \neq V(G)$. Further, $r < n$ (non-triviality of G) and $(V(G), \mathcal{B})$ is an $(n, t, n-r)$-design, with $b = |V(G)| = n$. Since $t \geq 3$, by Theorem 2, for each $v \in V(G)$ there is a $u \in V(G)$ such that $B(v) = V(G) \backslash \{u\}$. Thus G is a matching, and is thus trivially uniformly (t,r)-regular, after all.

References

[1] T. Beth, D. Jungnickel, and H. Lenz, *Design Theory*, Second Edition, Cambridge University Press, 1999.

[2] R. Faudree and D. Knisley, The characterization of large $(2,r)$-regular graphs, *Congressus Numerantium* 121 (1996), 105-108.

[3] T. Haynes and D. Knisley, Generalized maximum degree and totally regular graphs, *Utilitas Mathematica* 54 (1998), 211-221.

[4] Teresa W. Haynes and Lisa R. Markus, Generalized maximum degree, *Utilitas Mathematica* 59 (2001), 155-165.

[5] R. E. Jamison and P. D. Johnson Jr., The structure of (t,r)-regular graphs of large order, *Discrete Math.* 272 (2003), 297-300.

[6] Robert Jamison, Peter Johnson, Lisa Markus, Evan Morgan, and Emine Yazici, Strong (t,r)-regularity, submitted.

[7] P. D. Johnson Jr. and E. J. Morgan, Sharpening the Faudree-Knisley theorem on $(2,r)$-regularity, *Bulletin of the ICA* 29 (2003), 21-26.

[8] A Khodkar and C. D. Leach, On $(2,r)$-regular graphs, to appear in the *Bulletin of the ICA*.

[9] Abdollah Khodkar, David Leach, and David Robinson, Every $(2,r)$-regular graph is strongly regular, submitted.

[10] Douglas B. West, *Introduction to Graph Theory*, Second Edition, Prentice Hall, 2001.