On the Spectra of Totally Magic Labelings

J. P. McSorley and W. D. Wallis
Department of Mathematics
Southern Illinois University
Carbondale, IL 62901-4408, USA

Abstract

A *totally magic labeling* of a graph with v vertices and e edges is a one-to-one map taking the vertices and edges onto the integers $1, 2, \cdots, v+e$, such that the sum h of the label on a vertex and the labels on its incident edges is a constant independent of the choice of vertex, and the sum k of an edge label and the labels of the endpoints of the edge is constant. Such graphs appear to be rare. In this paper we examine the possible labelings of a union of an odd number of triangles, and determine the spectrum of possible values h, k for all known totally magic graphs.

Introduction

All graphs in this paper are finite, simple and undirected. Unless otherwise specified, the graph G has vertex set $V = V(G)$ and edge set $E = E(G)$ and we write e for $|E|$ and v for $|V|$.

A *totally magic labeling* λ on a graph G is a one-to-one map λ from $V(G) \cup E(G)$ onto the integers $1, 2, \ldots, v + e$, with the property that, given any vertex x,

$$\lambda(x) + \sum_{y \sim x} \lambda(xy) = h$$

for some constant h, where the sum is over all vertices y adjacent to x, and given any edge xy,

$$\lambda(x) + \lambda(xy) + \lambda(y) = k$$

for some constant k. A graph having a totally magic labeling is called a *totally magic graph.*

Totally magic labelings have been discussed in [1]. It is shown in that paper that totally magic graphs are very rare. The only known infinite families consist of the unions of an odd number of triangles, mK_3, where m is odd, and the same graphs with precisely one edge deleted. (On the other hand, the members of these families with m even are never totally magic.)

We define the *spectrum* of a totally magic graph to be the set of pairs of values (h, k) that arise as vertex and edge constants in totally magic labelings of the graph. We shall determine the spectra of the members of the above two infinite families, showing that there is a labeling for every h and k not excluded by simple arithmetical considerations.

Constructions

In this paper we examine labelings of mK_3 is detail. (This extends the discussion in Section 5 of [1].) Suppose λ is a totally magic labeling of mK_3, with vertex and edge constants h and k. We know that m is odd. Suppose the vertices of one of the triangles are labeled x, y, z; suppose the edge opposite the vertex labeled x receives the label X, and so on. Then

$$h = x + Y + Z = X + y + Z = X + Y + z, \tag{1}$$

$$k = X + y + z = x + Y + z = x + y + Z. \tag{2}$$

If we write $s = x + y + z$ and $S = X + Y + Z$, then (1) and (2) yield

$$\begin{aligned} 3h &= 2S + s, \\ 3k &= S + 2s, \\ h + k &= S + s. \end{aligned}$$

So

$$3h - 3k \;=\; S - s,$$

whence

$$\begin{aligned} S &= 2h - k, \\ s &= 2k - h. \end{aligned}$$

Finally, from $X + Y + Z = S$ and $x + Y + Z = h$ we obtain

$$X - x = S - h = h - k \tag{3}$$

for every choice of x, that is the difference between an edge label and the opposite vertex label is a constant, $h - k$.

Let us write d for $h - k$. It is clear that h and k determine d. On the other hand, if d is known, then h and k are determined. For each triangle,$S + s = h + k$, so summing over all triangles we get $m(h + k)$. But this is the sum of all vertex and edge labels in mK_3, so it equals $\frac{1}{2}6m(6m + 1)$, and $h + k = 3(6m + 1)$. Therefore

$$h = 9m + \tfrac{1}{2}(3 + d) \text{ and } k = 9m + \tfrac{1}{2}(3 - d). \tag{4}$$

Without loss of generality, let us assume $k < h$, that is $d = h - k > 0$. ($k = h$ is impossible, as it would imply $x = X$, and if $k > h$ then we can obtain a *dual* labeling with $k < h$ by interchanging the labels on each vertex and its opposite edge.) From (3), every edge label is at least $d + 1$, and $1, 2, \ldots, d$ are all vertex labels. The edges opposite those vertices receive labels $d + 1, d + 2, \ldots, 2d$. Proceeding in this way, we see that the set of vertex labels is

$$T_d(m) = \{1, 2, \ldots, d, 2d + 1, 2d + 2, \ldots, 3d, 4d + 1, \ldots, 6m - d\}$$

and the edge labels are

$$\{d+1, d+2, \dots, 2d, 3d+1, 3d+2, \dots, 4d, 5d+1, \dots, 6m\}.$$

So $6m$ is a multiple of $2d$. We have

Theorem 1 *If there is a totally magic labeling of mK_3, then m is odd and the vertex and edge constants have the values*

$$h = 9m + \tfrac{1}{2}(3+d) \text{ and } k = 9m + \tfrac{1}{2}(3-d), \tag{4}$$

for some divisor d of $3m$.

(The negative divisors d correspond to the labelings that are duals of those with positive d.)

Suppose d is a specified positive divisor of $3m$. Let $3m = ad$. Then the sum of the elements of $T_d(m)$ is

$$\begin{aligned} ms &= \sum_{i=1}^{d} \sum_{j=0}^{a-1} (2jd + i) \\ &= \sum_{i=1}^{d} (a(a-1)d + ai) \\ &= a(a-1)d^2 + \tfrac{1}{2}ad(d+1), \\ s &= 3(a-1)d + \tfrac{3}{2}(d+1). \end{aligned} \tag{5}$$

So the triples of vertex labels of triangles must form a partition of $T_d(m)$ into m triples, each with sum (5). Conversely, any such partition will define a totally magic labeling with the constants (4).

Lemma 2 *If m is an odd positive integer and d is any divisor of $3m$, then there exists a partition of*

$$\begin{aligned} T_d(m) &= \{1, 2, \dots, d, 2d+1, 2d+2, \dots, 3d, 4d+1, \dots, \\ &\qquad (2a-2)d+1, (2a-2)d+2, \dots, (2a-1)d\} \end{aligned}$$

into m triples, each with sum (5), where $a = 3m/d$.

Proof. We use two families of $3 \times n$ arrays, n odd, which are denoted A_n and B_n, and are defined as follows. A_n has j-th column $(j, n+2-2j, \frac{1}{2}(n-1)+j)$, where entries are reduced modulo n, if necessary, so that they lie in the range $1, 2, \ldots, n$. This array was used by Kotzig [2]. Each row is a permutation of $\{1, 2, \ldots, n\}$, and each column sums to $\frac{1}{2}(3n+3)$. B_n is constructed from a copy of A_n by adding n to every entry in the second row and $2n$ to every entry in the third row, so each column has sum $\frac{1}{2}(9n+3)$.

First, suppose d is a multiple of 3. Form a master array M by subtracting 1 from each entry of A_a and multiplying by $2d$. From each column of M we construct $d/3$ triples by adding each of the columns of $B_{d/3}$. It is easy to check that the $a(d/3) = m$ triples form a partition of $T_d(m)$. All the triples have the same sum, because both component arrays have constant column sum.

If d is not a multiple of 3, then a must be. In that case the master array is formed from $B_{a/3}$, again subtracting 1 and multiplying by $2d$. The columns of A_d are added. $\square$

Here is an illustration of the two constructions for the graph $15K_3$.

$$M = \begin{bmatrix} 0 & 18 & 36 & 54 & 72 \\ 72 & 36 & 0 & 54 & 18 \\ 36 & 54 & 72 & 0 & 18 \end{bmatrix} \quad B_{d/3} = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 4 & 5 \\ 8 & 9 & 7 \end{bmatrix}$$

Partition

1	2	3	19	20	21	37	38	39	55	56	57	73	74	75
78	76	77	42	40	41	6	4	5	60	58	59	24	22	23
44	45	43	62	63	61	80	81	79	8	9	7	26	27	25

Example for $a = 5, d = 9, m = 15$, column sum 123.

$$M = \begin{bmatrix} 0 & 10 & 20 \\ 50 & 30 & 40 \\ 70 & 80 & 60 \end{bmatrix} \quad A_d = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 4 & 2 \\ 3 & 4 & 5 & 1 & 2 \end{bmatrix}$$

Partition

1	2	3	4	5	11	12	13	14	15	21	22	23	24	25
55	53	51	54	52	35	33	31	34	32	45	43	41	44	42
73	74	75	71	72	83	84	85	81	82	63	64	65	61	62

Example for $a = 9, d = 5, m = 15$, column sum 129.

Theorem 3 *There is a totally magic labeling of mK_3 with vertex and edge constants h and k if and only if m is odd and $h = 9m + \frac{1}{2}(3 + d)$ and $k = 9m + \frac{1}{2}(3 - d)$, where d is a divisor of $3m$.*

Suppose λ is a totally magic labeling of mK_3 corresponding to a negative divisor d of $3m$, with vertex and edge constants h and k, and suppose edge u satisfies $\lambda(u) = 1$. Define $\mu(x) = \lambda(x) - 1$ for each edge or vertex x. Delete edge u. Then μ is a totally magic labeling of $P_3 \cup (m-1)K_3$ with vertex and edge constants $h-3$ and $k-3$. This construction can be reversed, so totally magic labelings of $P_3 \cup (m-1)K_3$ are equivalent to totally magic labelings of mK_3 in which 1 is an edge label — precisely those for which the divisor d is negative. Writing $f = -d$ (to avoid negative divisors), we have

Theorem 4 *There is a totally magic labeling of $P_3 \cup (m-1)K_3$ with vertex and edge constants h and k if and only if m is odd and $h = 9m - \frac{1}{2}(f + 3)$ and $k = 9m + \frac{1}{2}(f - 3)$, where f is a positive divisor of $3m$.*

Conclusion

The only known totally magic graphs not covered by Theorems 3 and 4 are the trivial case K_1, for which $h = 1$ and k is not defined, and $K_1 \cup P_3$, whose unique totally magic labeling has $(h, k) = (6, 9)$ (see [1]). Thus we have completely determined the spectrum of every known totally magic graph.

References

[1] G. Exoo, A. C. H. Ling, J. P. McSorley, N. C. K. Phillips and W. D. Wallis, Totally magic graphs. (*to appear*).

[2] A. Kotzig, *On magic valuations of trichromatic graphs. Reports of the CRM* **CRM–148**, December 1971.