Rhombic tilings of (n, k)-Ovals, (n, k, λ)-cyclic difference sets, and related topics

John P. McSorley*, Alan H. Schoen
Department of Mathematics
Mailcode 4408
Southern Illinois University
Carbondale, IL 62901-4408
mcsorley60@hotmail.com
alan_schoen@frontier.com
* Corresponding author

Abstract

Each fixed integer n has associated with it $\lfloor \frac{n}{2} \rfloor$ rhombs: $\rho_1, \rho_2, \ldots, \rho_{\lfloor \frac{n}{2} \rfloor}$, where, for each $1 \leq h \leq \lfloor \frac{n}{2} \rfloor$, rhomb ρ_h is a parallelogram with all sides of unit length and with smaller face angle equal to $h \times \frac{\pi}{n}$ radians.

An Oval is a centro-symmetric convex polygon all of whose sides are of unit length, and each of whose turning angles equals $\ell \times \frac{\pi}{n}$ for some positive integer ℓ. An (n, k)-Oval is an Oval with $2k$ sides tiled with rhombs $\rho_1, \rho_2, \ldots, \rho_{\lfloor \frac{n}{2} \rfloor}$; it is defined by its Turning Angle Index Sequence, a k-composition of n. For any fixed pair (n, k) we count and generate all (n, k)-Ovals up to translations and rotations, and, using multipliers, we count and generate all (n, k)-Ovals up to congruency. For odd n if an (n, k)-Oval contains a fixed number λ of each type of rhomb $\rho_1, \rho_2, \ldots, \rho_{\lfloor \frac{n}{2} \rfloor}$ then it is called a magic (n, k, λ)-Oval. We prove that a magic (n, k, λ)-Oval is equivalent to a (n, k, λ)-Cyclic Difference Set. For even n we prove a similar result. Using tables of Cyclic Difference Sets we find all magic (n, k, λ)-Ovals up to congruency for $n \leq 40$.

Many related topics including lists of (n, k)-Ovals, partitions of the regular $2n$-gon into Ovals, Cyclic Difference Families, partitions of triangle numbers, u-equivalence of (n, k)-Ovals, etc., are also considered.

Keywords: rhomb; tiling; polygon; oval; cyclic difference set; multiplier.

1 Introduction

An (n,k)-Oval is a centro-symmetric convex polygon all of whose sides are of unit length, and which is tiled by rhombs; see p.141 of Ball and Coxeter [1] and Section 3.1 of Schoen [8]. In this paper we investigate (n,k)-Ovals; it appears that this is the first significant piece of research concerning (n,k)-Ovals to be published in the mathematical literature. A preliminary version of some of this research first appeared in Schoen [8]. See Fig. 1 for an example of a $(15,6)$-Oval.

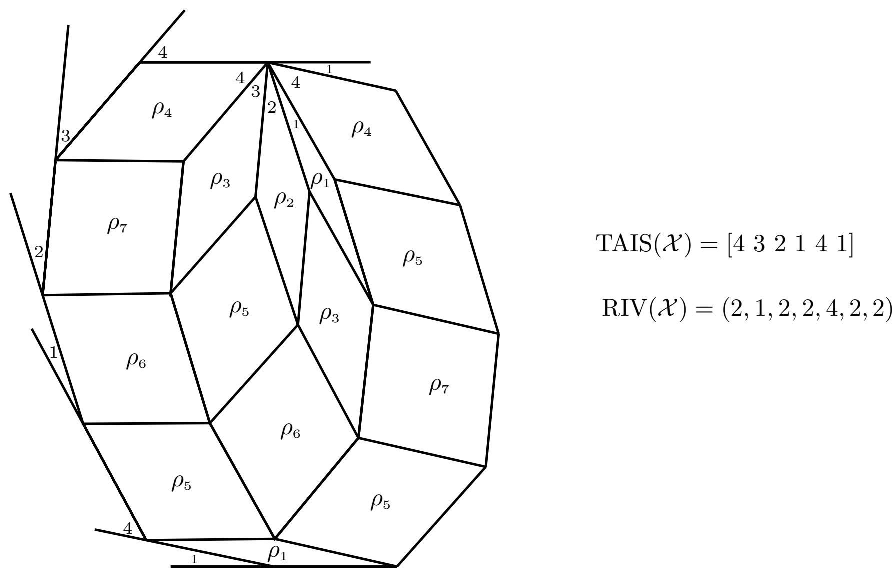


Figure 1: A $(15,6)$-Oval, $\mathcal{X}$, its TAIS and RIV.

In Section 2 of this paper we define an (n,k)-Oval using its Turning Angle Index Sequence (TAIS); we count all (n,k)-Ovals equivalent up to translations and rotations. We introduce the concept of a multiplier for an (n,k)-Oval and show how to generate all (n,k)-Ovals using multipliers.

In Section 3 we show the geometrical meaning of multiplier -1 for an (n,k)-Oval. We count those (n,k)-Ovals with multiplier -1, and those without multiplier -1. We define congruency for (n,k)-Ovals and count (n,k)-Ovals up to congruency.

In Section 4 we define the Rhombic Inventory Vector (RIV) of an (n, k)-Oval. This vector contains the number of each type of rhomb that an (n, k)-Oval contains. For each $2 \leq n \leq 10$ we list all (n, k)-Ovals up to congruency, and compute their RIVs.

In Section 5 we study magic (n, k, λ)-Ovals. For odd n a magic (n, k, λ)-Oval contains a fixed number $\lambda \geq 1$ of each type of rhomb $\rho_1, \rho_2, \ldots, \rho_{\lfloor \frac{n}{2} \rfloor}$; there is a similar definition for even n. We prove that a magic (n, k, λ)-Oval is equivalent to a (n, k, λ)-Cyclic Difference Set. Using tables of Cyclic Difference Sets we find all non-trivial magic (n, k, λ)-Ovals up to congruency for $n \leq 40$.

In Section 6 the rhombs of the regular $2n$-gon are partitioned into Ovals. Cyclic Difference Families are introduced and are shown to be equivalent to various Oval partitions; we also consider relevant integer partitions of the triangular number $\binom{n}{2}$.

In Section 7 we define u-equivalence for (n, k)-Ovals. The RIV's of two u-equivalent (n, k)-Ovals are closely related to each other. For each $2 \leq n \leq 10$ we list all (n, k)-Ovals up to u-equivalence .

2 (n, k)-Ovals, TAIS, the number of (n, k)-Ovals, multipliers, generating all (n, k)-Ovals

Each fixed integer $n \geq 2$ has associated with it $\lfloor \frac{n}{2} \rfloor$ *rhombs*: $\rho_1, \rho_2, \ldots, \rho_{\lfloor \frac{n}{2} \rfloor}$. For each $1 \leq h \leq \lfloor \frac{n}{2} \rfloor$ rhomb ρ_h is a parallelogram with all sides of unit length and with smaller face angle equal to $h \times \frac{\pi}{n}$ radians; h is the *principal index* of the rhomb. The index of an adjacent face angle is $n - h$. The 7 rhombs for $n = 15$ are shown in Fig. 2.

Definitions 2.1 Centro-symmetric, turning angle, Oval

(1) A polygon is *centro-symmetric* if it is unchanged by a rotation of π radians (half a circle).

(2) The *turning angle* at a vertex of a polygon is the supplement of the interior angle at that vertex.

(3) An *Oval* is a centro-symmetric convex polygon all of whose sides are of unit length, and each of whose turning angles equals $\ell \times \frac{\pi}{n}$ for some positive integer ℓ.

Every Oval necessarily has an even number of sides, which are arranged in k parallel pairs.

Definitions 2.2 (n, k)-Oval, Turning Angle Index Sequence – TAIS

(1) An *(n,k)-Oval* is an Oval with $2k$ sides; it is described by the pair (n, k) and by its

(2) *Turning Angle Index Sequence* (TAIS), a list of the turning angle indices for any k consecutive vertices.

We denote an arbitrary (n, k)-Oval by $\mathcal{O}$ and specify a *stem* vertex of $\mathcal{O}$; the TAIS of $\mathcal{O}$ is then the list of turning angle indices at the k consecutive vertices taken in a counter-clockwise direction starting from the first vertex after the stem vertex.

Remark 2.3 The TAIS T of an (n, k)-Oval is simply a k-composition of n, *i.e.*, an ordered list of k positive integers that sum to n: $T = [t_1\ t_2\ \cdots\ t_k]$ with each $t_i \geq 1$ and $\sum_{i=1}^{k} t_i = n$.

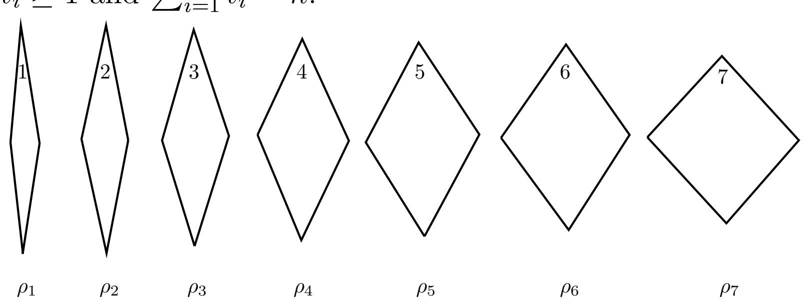


Figure 2: The 7 rhombs, and their principal indices, corresponding to $n = 15$.

Example 2.4 The regular $2n$-gon, $\{2n\}$, is an (n,n)-Oval with TAIS=$[\underbrace{1\ 1\ \cdots\ 1}_{n}]$. See Fig. 5 for a picture of the regular 12-gon, $\{12\}$.

Example 2.5 $(n,k) = (15,6)$. In Fig. 3(a) we show the $(15,6)$-Oval $\mathcal{X}$ with TAIS $T = [4\ 3\ 2\ 1\ 4\ 1]$. We write $\mathcal{X} = \mathcal{O}(T) = \mathcal{O}([4\ 3\ 2\ 1\ 4\ 1])$. In (b) the turning angle index at each vertex of $\mathcal{X}$ is shown, as well as all indices of the $\binom{6}{2} = 15$ rhombs in $\mathcal{X}$. Note that the indices along the straight line at an 'external' vertex sum to $n = 15$, and the indices around an 'internal' vertex sum to $2n = 30$.

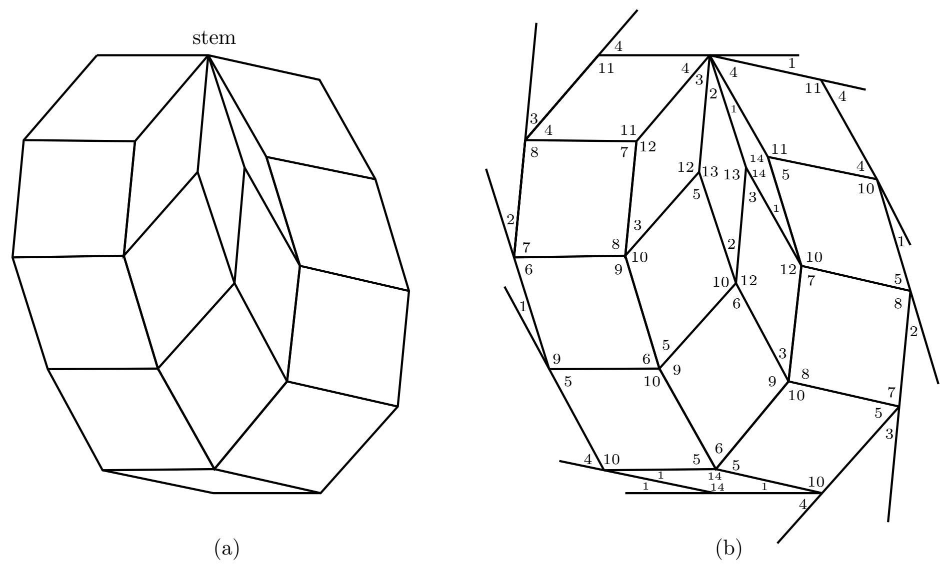


Figure 3: See Fig. 1. The $(15,6)$-Oval $\mathcal{X}$ with TAIS $T = [4\ 3\ 2\ 1\ 4\ 1]$.

Let $S = \{s_1, s_2, \ldots, s_k\}$ where $0 \le s_1 < s_2 < \cdots < s_k$ be a k-subset of $\mathbb{Z}_n$ with increasing elements. Throughout this paper the elements of S will always be written in increasing order.

Let $U(n)$ denote the group of units modulo n, *i.e.*, the multiplicative group of elements relatively prime to n.

Definitions 2.6 $uS+z$, $z-$equivalent and $\equiv_z$, cyclically-equivalent and $\equiv_{\text{cyc}}$

(1) $uS + z = \{us_1 + z, us_2 + z, \ldots, us_k + z\} \subseteq \mathbb{Z}_n$ for $u \in U(n)$ and $z \in \mathbb{Z}_n$.

(2) Two k-subsets S and S' of $\mathbb{Z}_n$ are *z-equivalent*, $S \equiv_z S'$, if there exists $z \in \mathbb{Z}_n$ such that $S = S' + z$.

(3) Two TAIS's T and T' are *cyclically-equivalent*, $T \equiv_{\text{cyc}} T'$, if T' is a cyclic permutation of T.

Remark 2.7 As an example of (3) above:

$$[t_1\ t_2\ t_3\ t_4] \equiv_{\text{cyc}} [t_4\ t_1\ t_2\ t_3] \equiv_{\text{cyc}} [t_3\ t_4\ t_1\ t_2] \equiv_{\text{cyc}} [t_2\ t_3\ t_4\ t_1].$$

Sometimes we use $=$ in place of $\equiv_z$ or $\equiv_{\text{cyc}}$ for convenience.

Let $\mathcal{S}^*(n, k)$ denote the set of all k-subsets $S = \{s_1, s_2, \ldots, s_k\} \subseteq \mathbb{Z}_n$ where $0 \le s_1 < s_2 < \cdots < s_k$. Then $\equiv_z$ is an equivalence relation on $\mathcal{S}^*(n, k)$. We denote the set of equivalences classes of $\equiv_z$ by $\mathcal{S}^*_{\equiv_z}(n, k)$. In an equivalence class $[S]_{\equiv_z}$ or $[S]$ we often use as representative the lowest member of $[S]$ in lexicographic ordering.

Let $\mathcal{T}^*(n, k)$ denote the set of all k-compositions of n, *i.e.*, the set of TAIS T for all (n, k)-Ovals. Then $\equiv_{\text{cyc}}$ is an equivalence relation on $\mathcal{T}^*(n, k)$. We denote the set of equivalences classes of $\equiv_{\text{cyc}}$ by $\mathcal{T}^*_{\equiv_{\text{cyc}}}(n, k)$, and a typical equivalences class by $[T]_{\equiv_{\text{cyc}}}$ or $[T]$.

Theorem 2.12 below gives a bijection between the sets $\mathcal{S}^*_{\equiv_z}(n, k)$ and $\mathcal{T}^*_{\equiv_{\text{cyc}}}(n, k)$.

Definitions 2.8 $\alpha(S)$ and $\mathcal{O}(\alpha(S))$ or $\mathcal{O}(T)$, $\beta(T)$

Let $S = \{s_1, s_2, \ldots, s_k\} \subseteq \mathbb{Z}_n$ where $0 \le s_1 < s_2 < \cdots < s_k$.

(1) $\alpha(S)$ is the ordered k-tuple

$$\alpha(S) = [s_2 - s_1, s_3 - s_2, \ldots, s_k - s_{k-1}, s_1 - s_k],$$

(note that $s_1 - s_k$ will be negative, it must be replaced with $n - s_1 + s_k$). Then $\mathcal{O}(\alpha(S)) = \mathcal{O}(T)$ is the (n, k)-Oval with TAIS $\alpha(S) = T$.

Let $T = [t_1\ t_2 \cdots t_k]$ be the TAIS of an (n, k)-Oval.

(2) $\beta(T)$ is the increasing k-subset of $\mathbb{Z}_n$

$$\beta(T) = \beta([t_1\ t_2 \cdots t_k]) = \{0, t_1, t_1 + t_2, \ldots, t_1 + t_2 + \cdots + t_{k-1}\}.$$

Remark 2.9 See similar definitions on p.221 of Beth, Jungnickel, and Lenz [3].

Example 2.10 $(n, k) = (15, 6)$. For the $(15, 6)$-Oval $\mathcal{X}$ of Example 2.5 with TAIS $T = [4\ 3\ 2\ 1\ 4\ 1]$ we have $X = S = \beta(T) = \{0, 4, 7, 9, 10, 14\}$, then $\alpha(X) = T$.

Compare the following Theorem with Lemma 9.8, p.221 of [3].

Theorem 2.11 *Let S and S' be k-subsets of $\mathbb{Z}_n$. Then $S \equiv_z S'$ if and only if $\alpha(S) \equiv_{\text{cyc}} \alpha(S')$.*

Proof. Necessity: as usual let $S = \{s_1, s_2, \ldots, s_k\}$ where $0 \leq s_1 < s_2 < \ldots < s_k$ and $\alpha(S) = [s_2 - s_1, \ldots, s_k - s_{k-1}, s_1 - s_k]$. Suppose $S \equiv_z S'$ then there exists $z \in \mathbb{Z}_n$ with

$$\begin{aligned} S' = S + z &= \{s_1 + z, s_2 + z, \ldots, s_k + z\} \\ &= \{s_i + z, s_{i+1} + z, \ldots, s_k + z, s_1 + z, s_2 + z, \ldots, s_{i-1} + z\} \end{aligned}$$

where $0 \leq s_i + z < s_{i+1} + z < \ldots < s_{i-1} + z$ is an increasing sequence for some $i = 1, 2, \ldots, k$. So

$$\begin{aligned} \alpha(S') &= [s_{i+1} - s_i, \ldots, s_1 - s_k, s_2 - s_1, \ldots, s_{i-1} - s_{i-2}, s_i - s_{i-1}] \\ &\equiv_{\text{cyc}} [s_2 - s_1, \ldots, s_{i-1} - s_{i-2}, s_i - s_{i-1}, s_{i+1} - s_i, \ldots, s_1 - s_k] \\ &= \alpha(S), \text{ as required.} \end{aligned}$$

Sufficiency: if $\alpha(S) \equiv_{\rm cyc} \alpha(S')$ then $\alpha(S')$ is a cyclic permutation of $\alpha(S)$. Without loss of generality let $\alpha(S) = [t_1\, t_2\, \cdots\, t_k]$ and $\alpha(S') = [t_i\, t_{i+1}\, \cdots\, t_k\, t_1\, \cdots\, t_{i-1}]$ for some $i = 1, 2, \ldots, k$. Then $\beta(\alpha(S)) = \{0, t_1, t_1 + t_2, \ldots, t_1 + \cdots + t_{k-1}\}$ and

$$\begin{aligned}\beta(\alpha(S')) &= \{0, t_i, t_i + t_{i+1}, \ldots, t_i + \cdots + t_k + t_1 + \cdots + t_{i-2}\}\\ &= \beta(\alpha(S)) + (t_i + \cdots + t_k)\\ &\equiv_z \beta(\alpha(S)).\end{aligned}$$

So $\beta(\alpha(S')) \equiv_z \beta(\alpha(S))$, but from Definitions 2.8 we have $\beta(\alpha(S)) = S - s_1 \equiv_z S$ for any S, and so $S \equiv_z S'$ as required. □

Theorem 2.12 *Let $\alpha_{\equiv} : \mathcal{S}^*_{\equiv_z}(n, k) \leftrightarrow \mathcal{T}^*_{\equiv_{\rm cyc}}(n, k)$ be given by $\alpha_{\equiv}([S]) \leftrightarrow [\alpha(S)]$. Then $\alpha_{\equiv}$ is a bijection, and $|\mathcal{S}^*_{\equiv_z}(n, k)| = |\mathcal{T}^*_{\equiv_{\rm cyc}}(n, k)|$.*

Remark 2.13 Geometrically speaking, if two TAIS's T and T' are cyclically-equivalent, then the Ovals $\mathcal{O}(T)$ and $\mathcal{O}(T')$ can be 'moved' to one another in the plane using translations and rotations, a reflection is not required; we write $\mathcal{O}(T) = \mathcal{O}(T')$. The converse is also true. Thus $T \equiv_{\rm cyc} T'$ if and only if $\mathcal{O}(T) = \mathcal{O}(T')$.

Definitions 2.14 $\mathcal{O}^*(n, k)$, $\mathcal{O}(n, k)$

(1) $\mathcal{O}^*(n, k)$ is the set of (n, k)-Ovals equivalent up to translations and rotations.

(2) $\mathcal{O}(n, k) = |\mathcal{O}^*(n, k)|$ is the number of (n, k)-Ovals equivalent up to translations and rotations.

Each Oval in $\mathcal{O}^*(n, k)$ has associated with it an equivalence class $[T]$ in $\mathcal{T}^*_{\equiv_{\rm cyc}}(n, k)$, and conversely each equivalence class $[T]$ in $\mathcal{T}^*_{\equiv_{\rm cyc}}(n, k)$ gives an Oval $\mathcal{O}(T)$ in $\mathcal{O}^*(n, k)$. So $\mathcal{O}(n, k) = |\mathcal{T}^*_{\equiv_{\rm cyc}}(n, k)|$. This function is well-known to be the number of necklaces of size n with k white and $n - k$ black beads; for an explicit calculation of $\mathcal{O}(n, k)$ see p.468 of Van Lint and Wilson [10]. Thus, letting $\gcd(n, k)$ denote the greatest common divisor of n and k, and $\phi(x)$ denote Euler's totient function, we have the following.

Theorem 2.15 *For $n \geq 2$ and $k \geq 2$, the number of (n, k)-Ovals is*

$$\mathcal{O}(n, k) = \frac{1}{n} \sum_{d | \gcd(n,k)} \phi(d) \binom{\frac{n}{d}}{\frac{k}{d}}. \qquad (1)$$

2.1 Multipliers, generating all (n,k)-Ovals

We wish to generate all Ovals in $\mathcal{O}^*(n,k)$. To do this we find a representative of each equivalence class $[S]$ in $\mathcal{S}^*_{\equiv_z}(n,k)$ and then use Theorem 2.12 to find a representative of each equivalence class $[T]$ in $\mathcal{T}^*_{\equiv_{\rm cyc}}(n,k)$.

Definitions 2.16 multiplier m and mult(S), mult$(\mathcal{O})$

Let S be a k-subset of $\mathbb{Z}_n$:

(1) $m \in U(n)$ is a *multiplier* of S if $S \equiv_z mS$, *i.e.*, if there exists $z \in \mathbb{Z}_n$ with $S = mS + z$. The set of multipliers of S is mult(S).

Let $\mathcal{O}(T)$ be a (n,k)-Oval with TAIS T:

(2) $m \in U(n)$ is a *multiplier* of $\mathcal{O}(T)$ if m is a multiplier of $S = \beta(T)$. The set of multipliers of $\mathcal{O}(T)$ is mult$(\mathcal{O}(T))$ = mult(S).

Remark 2.17 See Chapter VI of [3] for examples of how multipliers are used in the theory of Cyclic Difference Sets; see also Section 5 of this paper. The set mult(S) is a subgroup of $U(n)$, and if $S \equiv_z S'$ then mult(S) = mult(S'). Let T and T' be two different TAIS of an (n,k)-Oval $\mathcal{O}$. Then $T \equiv_{\rm cyc} T'$ and so $\beta(T) \equiv_z \beta(T')$ by Theorem 2.11, and then mult$(\beta(T))$ = mult$(\beta(T'))$. Hence mult$(\mathcal{O})$ is independent of the TAIS of $\mathcal{O}$.

Example 2.18 $(n,k) = (15,6)$. For the $(15,6)$-Oval $\mathcal{X}$ of Examples 2.5 and 2.10 we have $X = \{0,4,7,9,10,14\}$ and so mult$(\mathcal{X})$ = mult$(X) = \{1\}$, the trivial group. For an example of a 6-set of $\mathbb{Z}_{15}$ with non-trivial multiplier group consider $Y = \{0,1,4,7,10,13\}$, here mult$(Y) = \{1,4,7,13\}$.

Now $m \in$ mult(S) if and only if $S \equiv_z mS$. Hence the number of z-inequivalent sets in $\{uS : u \in U(n)\}$ equals the index of mult(S) in $U(n)$, *i.e.*, equals $|U(n) : \text{mult}(S)| = \frac{|U(n)|}{|\text{mult}(S)|}$.

As an example of how to generate all Ovals in $\mathcal{O}^*(n,k)$ we generate all Ovals in $\mathcal{O}^*(7,3)$.

We have $U(7) = \{1,2,3,4,5,6\}$ and so $|U(7)| = 6$.

Start with $A = \{0,1,2\}$. So mult$(A) = \{1,-1\}$ and $|U(7) : \text{mult}(A)| = 3$. The 3 cosets of mult$(A)$ in $U(7)$ are mult(A), $2\,$mult(A), and $3\,$mult(A). Hence the 3 z-inequivalent sets in $\{uA : u \in U(n)\}$ are $A_1 = A, A_2 = 2A = \{0,2,4\}$, and $A_3 = 3A = \{0,3,6\} \equiv_z \{0,1,4\}$.

Then choose $A' = \{0,1,3\}$ from $\mathcal{S}^*(7,3)\backslash([A_1] \cup [A_2] \cup [A_3])$. We have $\text{mult}(A') = \{1,2,4\}$ and $|U(7) : \text{mult}(A')| = 2$. The 2 cosets of $\text{mult}(A')$ in $U(7)$ are $\text{mult}(A')$ and $3\,\text{mult}(A')$. Hence the 2 z-inequivalent sets in $\{uA' : u \in U(n)\}$ are $A'_1 = A'$ and $A'_2 = 3A' = \{3,5,6\} \equiv_z \{0,1,5\}$.

Now $\mathcal{S}^*(7,3)\backslash([A_1] \cup [A_2] \cup [A_3] \cup [A'_1] \cup [A'_2]) = \emptyset$, so we stop. See Example 2.19.

Example 2.19 $(n,k) = (7,3)$. Equation (1) gives $\mathcal{O}(7,3) = |\mathcal{T}^*_{\equiv_{\text{cyc}}}(7,3)| = \frac{1}{7}\phi(1)\binom{7}{3} = 5$. Representatives of the 5 equivalence classes in both $\mathcal{S}^*_{\equiv_z}(7,3)$ and $\mathcal{T}^*_{\equiv_{\text{cyc}}}(7,3)$, and the bijection between them, are given in the table below. The 5 $(7,3)$-Ovals up to translations and rotations are $\mathcal{O}^*(7,3) = \{\mathcal{O}(T_1), \mathcal{O}(T_2), \mathcal{O}(T_3), \mathcal{O}(T_4), \mathcal{O}(T_5)\}$, see Fig. 4 below. We will see that multiplier -1 plays an important role in this paper. We use 'A_i' for a set with multiplier -1, and 'B_i' for a set without multiplier -1.

S		T	$\text{mult}(S)$	$\frac{\lvert U(7)\rvert}{\lvert\text{mult}(S)\rvert}$
$A_1 = \{0,1,2\}$	$\leftrightarrow$	$T_1 = [1\ 1\ 5]$	$\{1,-1\}$	3
$A_2 = \{0,2,4\}$	$\leftrightarrow$	$T_2 = [2\ 2\ 3]$	$\{1,-1\}$	
$A_3 = \{0,1,4\}$	$\leftrightarrow$	$T_3 = [1\ 3\ 3]$	$\{1,-1\}$	
$B_1 = \{0,1,3\}$	$\leftrightarrow$	$T_4 = [1\ 2\ 4]$	$\{1,2,4\}$	2
$B_2 = \{0,1,5\}$	$\leftrightarrow$	$T_5 = [1\ 4\ 2]$	$\{1,2,4\}$	

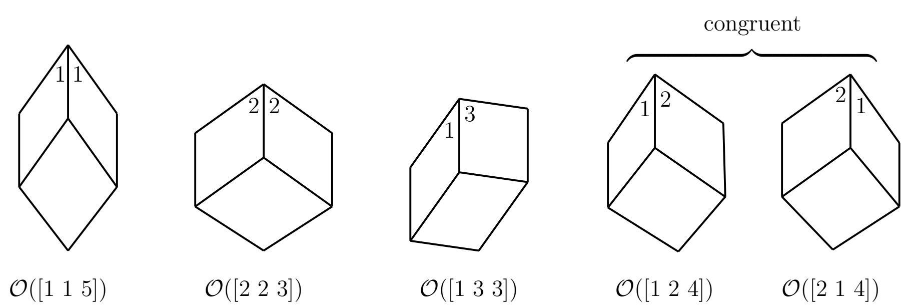


Figure 4: The $\mathcal{O}(7,3) = 5$ $(7,3)$-Ovals up to translations and rotations. The last 2 form a congruent enantiomorphic pair.

It is clear how to generalize Example 2.19 to generate all Ovals in $\mathcal{O}^*(n,k)$, *i.e.*, all (n,k)-Ovals up to translations and rotations, for an arbitrary (n,k) starting with $A = \{0, 1, \ldots, k-1\}$.

3 Multiplier -1, reversible T, congruent Ovals, various counts

In this Section we consider multiplier -1 of an (n,k)-Oval $\mathcal{O}$. We will return to consideration of multiplier -1 in Section 5.

Let $T = [t_1\ t_2\ \cdots\ t_k]$ be a TAIS of an (n,k)-Oval $\mathcal{O}$.

Definition 3.1 $\overleftarrow{T} = [t_k\ t_{k-1} \cdots\ t_1]$ is the *reverse* of T.

Lemma 3.2 *Let S and S' be k-subsets of $\mathbb{Z}_n$. Then*

(i) $\alpha(-S) \equiv_{\text{cyc}} \overleftarrow{\alpha(S)}$.

(ii) $S \equiv_z -S'$ *if and only if* $\alpha(S) \equiv_{\text{cyc}} \overleftarrow{\alpha(S')}$.

Proof. (i) Let $S = \{s_1, s_2, \ldots, s_k\}$, where $0 \leq s_1 < s_2 < \cdots < s_k$. Then $-S = \{-s_1, -s_2, \ldots, -s_k\} = \{n - s_1, n - s_2, \ldots, n - s_k\} = \{n - s_k, n - s_{k-1}, \ldots, n - s_2, n - s_1\}$, in increasing order. So $\alpha(-S) = [s_k - s_{k-1}, \ldots, s_2 - s_1, s_1 - s_k] \equiv_{\text{cyc}} [s_1 - s_k, s_k - s_{k-1}, \ldots, s_2 - s_1] = \overleftarrow{\alpha(S)}$.

(ii) Necessity: let $S \equiv_z -S'$ then $\alpha(S) \equiv_{\text{cyc}} \alpha(-S') \equiv_{\text{cyc}} \overleftarrow{\alpha(S')}$ using Theorem 2.11 and then part (i) above.

Sufficiency: let $\alpha(S) \equiv_{\text{cyc}} \overleftarrow{\alpha(S')}$ then $\alpha(S) \equiv_{\text{cyc}} \alpha(-S')$ by part (i) applied to S', and so $S \equiv_z -S'$ by Theorem 2.11. □

Definition 3.3 TAIS T is *reversible* if it is cyclically-equivalent to its reverse, *i.e.*, if $T \equiv_{\text{cyc}} \overleftarrow{T}$, (equivalently, $T \in \left[\overleftarrow{T}\right]$ or $\overleftarrow{T} \in [T]$).

Theorem 3.4 *Let S be a k-subset of $\mathbb{Z}_n$. Then $-1 \in \text{mult}(S)$ if and only if $\alpha(S)$ is reversible.*

Proof. Now $-1 \in \text{mult}(S)$ if and only if $S \equiv_z -S$, if and only if $\alpha(S) \equiv_{\text{cyc}} \overleftarrow{\alpha(S)}$, if and only if $\alpha(S)$ is reversible. □

Definitions 3.5 $\mathcal{O}(n,k;-1)$, $\mathcal{O}(n,k;\overline{-1})$

(1) $\mathcal{O}(n,k;-1)$ is the number of (n,k)-Ovals with -1 as a multiplier.

(2) $\mathcal{O}(n,k;\overline{-1})$ is the number of (n,k)-Ovals without -1 as a multiplier.

A *k-reverse* of n is a reversible k-composition of n. In McSorley [6] using Polya Theory we count the number of k-reverses of n up to cyclic permutation; this number is denoted by $\mathcal{R}_{\equiv}(n,k)$. From Theorem 3.4 above we have $\mathcal{O}(n,k;-1) = \mathcal{R}_{\equiv}(n,k)$.

Theorem 3.6 *For $n \geq 2$ and $k \geq 2$, the number of (n,k)-Ovals with -1 as a multiplier is*

$$\mathcal{O}(n,k;-1) = \begin{cases} \binom{\frac{n-2}{2}}{\frac{k-1}{2}}, & \textit{if } n \textit{ is even and } k \textit{ is odd;} \\ \binom{\frac{n-1}{2}}{\frac{k-1}{2}}, & \textit{if } n \textit{ is odd and } k \textit{ is odd;} \\ \binom{\frac{n}{2}}{\frac{k}{2}}, & \textit{if } n \textit{ is even and } k \textit{ is even;} \\ \binom{\frac{n-1}{2}}{\frac{k}{2}}, & \textit{if } n \textit{ is odd and } k \textit{ is even.} \end{cases}$$

For a given TAIS T we obtain Oval $\mathcal{O}\left(\overleftarrow{T}\right)$ from Oval $\mathcal{O}(T)$ by reflecting $\mathcal{O}(T)$ in a straight line that (for simplicity) does not intersect $\mathcal{O}(T)$. We denote the reflection of $\mathcal{O}$ by $\overleftarrow{\mathcal{O}}$.

When Ovals $\mathcal{O}(T)$ and $\mathcal{O}\left(\overleftarrow{T}\right)$ cannot be moved to one another using only translations and rotations, we say they are *enantiomorphs* of each other. In this case $\mathcal{O}(T) \neq \mathcal{O}\left(\overleftarrow{T}\right)$ and a reflection is required to move $\mathcal{O}(T)$ to $\mathcal{O}\left(\overleftarrow{T}\right)$ and vice-versa. (Oval $\mathcal{O}(T)$ is congruent to $\mathcal{O}\left(\overleftarrow{T}\right)$; see Section 3.1.) These comments and Theorem 3.4 give the following.

Theorem 3.7 *Let $\mathcal{O}(T)$ be an (n,k)-Oval.*

(i) $\mathcal{O}(T)$ has multiplier -1 if and only if T is reversible, if and only if $\mathcal{O}(T) = \mathcal{O}\left(\overleftarrow{T}\right)$.

(ii) $\mathcal{O}(T)$ does not have multiplier -1 if and only if T is not reversible, if and only if $\mathcal{O}(T) \neq \mathcal{O}\left(\overleftarrow{T}\right)$. Such Ovals occur in $\{\mathcal{O}(T), \mathcal{O}\left(\overleftarrow{T}\right)\}$ (congruent) enantiomorphic pairs in $\mathcal{O}^*(n,k)$. (Hence there is an even number of Ovals in $\mathcal{O}^*(n,k)$ without multiplier -1.)

Example 3.8 $(n,k) = (7,3)$. See Example 2.19.
$\mathcal{O}^*(7,3) = \{\mathcal{O}(T_1), \mathcal{O}(T_2), \mathcal{O}(T_3), \mathcal{O}(T_4), \mathcal{O}(T_5)\}$, and Theorem 3.6 gives $\mathcal{O}(7,3;-1) = \binom{3}{1} = 3$.

If $i = 1, 2$, or 3, then $-1 \in \text{mult}(\mathcal{O}(T_i))$ and so $T_i \equiv_{\text{cyc}} \overleftarrow{T_i}$; *eg.*, for $i = 1$ we have $[1\ 1\ 5] \equiv_{\text{cyc}} [5\ 1\ 1] (= \overleftarrow{[1\ 1\ 5]})$.

If $i = 4$, or 5, then $-1 \notin \text{mult}(\mathcal{O}(T_i))$ and so $T_i \not\equiv_{\text{cyc}} \overleftarrow{T_i}$; *eg.*, for $i = 4$ we have $[1\ 2\ 4] \not\equiv_{\text{cyc}} [4\ 2\ 1] (= \overleftarrow{[1\ 2\ 4]})$.

The pair $\{\mathcal{O}(T_4), \mathcal{O}(T_5)\} = \{\mathcal{O}(T_4), \mathcal{O}\left(\overleftarrow{T_4}\right)\}$ is a (congruent) enantiomorphic pair referred to in Theorem 3.7(ii).

3.1 Congruent Ovals

Definitions 3.9 congruent and $\equiv_{\text{c}}$

(1) Two k-subsets S and S' of $\mathbb{Z}_n$ are *congruent*, $S \equiv_{\text{c}} S'$, if $S \equiv_z S'$ or $S \equiv_z -S'$.

(2) Two TAIS T and T' are *congruent*, $T \equiv_{\text{c}} T'$, if $T \equiv_{\text{cyc}} T'$ or $T \equiv_{\text{cyc}} \overleftarrow{T'}$.

(3) Two (n,k)-Ovals $\mathcal{O}$ and $\mathcal{O}'$ are *congruent*, $\mathcal{O} \equiv_{\text{c}} \mathcal{O}'$, if $\mathcal{O} = \mathcal{O}'$ or $\mathcal{O} = \overleftarrow{\mathcal{O}'}$, *i.e.*, if $\mathcal{O}$ can be moved to $\mathcal{O}'$ by a sequence of translations, rotations, or reflections, (isometries).

Then, from Theorem 2.11 and Lemma 3.2, we have the following.

Theorem 3.10 *Let S and S' be k-subsets of $\mathbb{Z}_n$. Then $S \equiv_{\text{c}} S'$ if and only if $\alpha(S) \equiv_{\text{c}} \alpha(S')$, if and only if $\mathcal{O}(\alpha(S)) \equiv_{\text{c}} \mathcal{O}(\alpha(S'))$.*

Definition 3.11 $Mult(S) = \text{mult}(S) \cup -\text{mult}(S)$.

Remark 3.12 It is straightforward to show that $\text{Mult}(S)$ is a subgroup of $U(n)$. If $-1 \in \text{mult}(S)$ then $\text{Mult}(S) = \text{mult}(S)$, and if $-1 \notin \text{mult}(S)$ then $|\text{Mult}(S)| = 2\,|\text{mult}(S)|$.

Definitions 3.13 $\mathcal{O}^*_{\text{c}}(n,k)$, $\mathcal{O}_{\text{c}}(n,k)$

(1) $\mathcal{O}_c^*(n,k)$ is the set of (n,k)-Ovals up to congruency.

(2) $\mathcal{O}_c(n,k) = |\mathcal{O}_c^*(n,k)|$ is the number of (n,k)-Ovals up to congruency.

In order to generate the set $\mathcal{O}_c^*(n,k)$ for an arbitrary (n,k) we may use the procedure in Section 2.1 to find $\mathcal{O}^*(n,k)$ and then combine congruent enantiomorphic pairs of Ovals; see Theorem 3.7(ii). Alternatively, we may use this procedure with the group $\mathrm{mult}(S)$ replaced by $\mathrm{Mult}(S)$.

Example 3.14 $(n,k) = (7,3)$. See Examples 2.19 and 3.8.

To find $\mathcal{O}_c^*(7,3)$ using the first method mentioned above we start with $\mathcal{O}^*(7,3) = \{\mathcal{O}(T_1), \mathcal{O}(T_2), \mathcal{O}(T_3), \mathcal{O}(T_4), \mathcal{O}\big(\overleftarrow{T_4}\big)\}$ and combine the last 2 Ovals into a single congruency class to give $\mathcal{O}_c^*(7,3) = \{\mathcal{O}(T_1), \mathcal{O}(T_2), \mathcal{O}(T_3), \mathcal{O}(T_4)\}$.

Using the second method, the procedure of Section 2.1 with $\mathrm{mult}(S)$ replaced by $\mathrm{Mult}(S)$ gives the following table:

S		T	$\mathrm{Mult}(S)$	$\frac{\lvert U(7)\rvert}{\lvert\mathrm{Mult}(S)\rvert}$
$A_1 = \{0,1,2\}$	$\leftrightarrow$	$T_1 = [1\ 1\ 5]$	$\{1,-1\}$	3
$A_2 = \{0,2,4\}$	$\leftrightarrow$	$T_2 = [2\ 2\ 3]$	$\{1,-1\}$	
$A_3 = \{0,1,4\}$	$\leftrightarrow$	$T_3 = [1\ 3\ 3]$	$\{1,-1\}$	
$B_1 = \{0,1,3\}$	$\leftrightarrow$	$T_4 = [1\ 2\ 4]$	$U(7)$	1

This also gives $\mathcal{O}_c^*(7,3) = \{\mathcal{O}(T_1), \mathcal{O}(T_2), \mathcal{O}(T_3), \mathcal{O}(T_4)\}$, the set of all $(7,3)$-Ovals up to congruency.

3.2 $\mathcal{O}_c(n,k)$, $\mathcal{O}_c(n,k;-1)$, and $\mathcal{O}_c(n,k;\overline{-1})$

Definitions 3.15 $\mathcal{O}_c(n,k;-1)$, $\mathcal{O}_c(n,k;\overline{-1})$

(1) $\mathcal{O}_c(n,k;-1)$ is the number of (n,k)-Ovals with -1 as a multiplier, up to congruency.

(2) $\mathcal{O}_c(n,k;\overline{-1})$ is the number of (n,k)-Ovals without -1 as a multiplier, up to congruency.

Lemma 3.16

$$\mathcal{O}_c(n,k) = \frac{1}{2}\Big(\mathcal{O}(n,k) + \mathcal{O}(n,k;-1)\Big).$$

Proof.

$$\begin{aligned}
\mathcal{O}_{\rm c}(n,k) &= \mathcal{O}_{\rm c}(n,k;-1) + \mathcal{O}_{\rm c}(n,k;\overline{-1}) \\
&= \mathcal{O}(n,k;-1) + \frac{1}{2}\mathcal{O}(n,k;\overline{-1}) \\
&= \mathcal{O}(n,k;-1) + \frac{1}{2}(\mathcal{O}(n,k) - \mathcal{O}(n,k;-1)) \\
&= \frac{1}{2}(\mathcal{O}(n,k) + \mathcal{O}(n,k;-1)).
\end{aligned}$$

At the second line we use $\mathcal{O}(n,k;-1) = \mathcal{O}_{\rm c}(n,k;-1)$ because if $\mathcal{O}$ and $\mathcal{O}'$ both have -1 as a multiplier then, from Definitions 3.9(3) and Theorem 3.7(i), we have $\mathcal{O} = \mathcal{O}'$ if and only if $\mathcal{O} \equiv_{\rm c} \mathcal{O}'$. And $\mathcal{O}_{\rm c}(n,k;\overline{-1}) = \frac{1}{2}\mathcal{O}(n,k;\overline{-1})$ comes directly from Theorem 3.7(ii). □

Recall that $\mathcal{O}(n,k)$ is given explicitly in Equation (1).

Theorem 3.17 *For $n \geq 2$ and $k \geq 2$, the number of (n,k)-Ovals up to congruency is*

$$\mathcal{O}_{\rm c}(n,k) = \begin{cases}
\frac{1}{2}\Big(\mathcal{O}(n,k) + \binom{\frac{n-2}{2}}{\frac{k-1}{2}}\Big), & \text{if } n \text{ is even and } k \text{ is odd;} \\
\frac{1}{2}\big(\mathcal{O}(n,k) + \binom{\frac{n-1}{2}}{\frac{k-1}{2}}\big), & \text{if } n \text{ is odd and } k \text{ is odd;} \\
\frac{1}{2}\Big(\mathcal{O}(n,k) + \binom{\frac{n}{2}}{\frac{k}{2}}\Big), & \text{if } n \text{ is even and } k \text{ is even;} \\
\frac{1}{2}\Big(\mathcal{O}(n,k) + \binom{\frac{n-1}{2}}{\frac{k}{2}}\Big), & \text{if } n \text{ is odd and } k \text{ is even.}
\end{cases}$$

Theorem 3.6 now gives the following.

Theorem 3.18 *For $n \geq 2$ and $k \geq 2$, the number of (n,k)-Ovals without -1 as a multiplier up to congruency is*

$$\mathcal{O}_{\rm c}(n,k;\overline{-1}) = \begin{cases}
\frac{1}{2}\Big(\mathcal{O}(n,k) - \binom{\frac{n-2}{2}}{\frac{k-1}{2}}\Big), & \text{if } n \text{ is even and } k \text{ is odd;} \\
\frac{1}{2}\big(\mathcal{O}(n,k) - \binom{\frac{n-1}{2}}{\frac{k-1}{2}}\big), & \text{if } n \text{ is odd and } k \text{ is odd;} \\
\frac{1}{2}\Big(\mathcal{O}(n,k) - \binom{\frac{n}{2}}{\frac{k}{2}}\Big), & \text{if } n \text{ is even and } k \text{ is even;} \\
\frac{1}{2}\Big(\mathcal{O}(n,k) - \binom{\frac{n-1}{2}}{\frac{k}{2}}\Big), & \text{if } n \text{ is odd and } k \text{ is even.}
\end{cases}$$

$n\backslash k$	2	3	4	5	6	7	8	9	10	$\mathcal{O}_{\mathrm{c}}(n)$
2	1									1
3	1	1								2
4	2	1	1							4
5	2	2	1	1						6
6	3	3	3	1	1					11
7	3	4	4	3	1	1				16
8	4	5	8	5	4	1	1			28
9	4	7	10	10	7	4	1	1		44
10	5	8	16	16	16	8	5	1	1	76
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

(a) $\mathcal{O}_{\mathrm{c}}(n,k)$

$n\backslash k$	2	3	4	5	6	7	8	9	10	$\mathcal{O}_{\mathrm{c}}(n;-1)$
2	1									1
3	1	1								2
4	2	1	1							4
5	2	2	1	1						6
6	3	2	3	1	1					10
7	3	3	3	3	1	1				14
8	4	3	6	3	4	1	1			22
9	4	4	6	6	4	4	1	1		30
10	5	4	10	6	10	4	5	1	1	46
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

(b) $\mathcal{O}_{\mathrm{c}}(n,k;-1)$

$n\backslash k$	2	3	4	5	6	7	8	9	10	$\mathcal{O}_{\mathrm{c}}(n;\overline{-1})$
2	0									0
3	0	0								0
4	0	0	0							0
5	0	0	0	0						0
6	0	1	0	0	0					1
7	0	1	1	0	0	0				2
8	0	2	2	2	0	0	0			6
9	0	3	4	4	3	0	0	0		14
10	0	4	6	10	6	4	0	0	0	30
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

(c) $\mathcal{O}_{\mathrm{c}}(n,k;\overline{-1})$

Table 1: Values of $\mathcal{O}_{\mathrm{c}}(n,k)$, $\mathcal{O}_{\mathrm{c}}(n,k;-1)$, and $\mathcal{O}_{\mathrm{c}}(n,k;\overline{-1})$ for $2 \le k \le n \le 10$, and of $\mathcal{O}_{\mathrm{c}}(n)$, $\mathcal{O}_{\mathrm{c}}(n;-1)$, and $\mathcal{O}_{\mathrm{c}}(n;\overline{-1})$ for $2 \le n \le 10$.

See Table 1(a). The triangle of values of $\mathcal{O}_{\mathrm{c}}(n,k)$ when read row-by-row gives sequence A052307 in the Online Encyclopedia of Integer Sequences [7].

See Table 1(b). The triangle of values of $\mathcal{O}_{\mathrm{c}}(n,k;-1) = \mathcal{O}(n,k;-1)$ (see Theorem 3.6) is equal to the triangle of sequence A119963 in [7] (with the first two columns of 1's removed). So $\mathcal{O}_{\mathrm{c}}(n,k;-1)$ gives the *first* combinatorial interpretation of sequence A119963 in [7]. Thus (ignoring the first two columns of 1's) the (n,k) term in the triangle of sequence A119963 is the number of (n,k)-Ovals with -1 as a multiplier, up to congruency. For the sequence of row sums of the triangle of sequence A119963 see sequence A029744, and the comment 'Necklaces with n beads that are the same when turned over'.

See Table 1(c). When the triangle of values of $\mathcal{O}_c(n,k;\overline{-1})$ is read row-by-row we obtain a new sequence, see sequence A180472 in [7]. For the sequence of row sums of this triangle see sequence A059076: 'Number of orientable necklaces with n beads and two colors; *i.e.*, turning over the necklace does not leave it unchanged'.

Example 3.19 $(n,k) = (7,3)$. From Example 3.14 the number of $(7,3)$-Ovals up to congruency is 4. Theorem 3.17 gives $\mathcal{O}_c(7,3) = \frac{1}{2}(\mathcal{O}(7,3)+\binom{3}{1}) = \frac{1}{2}(5+3) = 4$, also. Of these 4 Ovals, 3 have -1 as a multiplier, and 1 does not. Theorem 3.6 gives $\mathcal{O}_c(7,3;-1) = \binom{3}{1} = 3$, and Theorem 3.18 gives $\mathcal{O}_c(7,3;\overline{-1}) = \frac{1}{2}(\mathcal{O}(7,3) - \binom{3}{1}) = \frac{1}{2}(5-3) = 1$. Thus all counts for $(n,k) = (7,3)$ from Example 3.14 are confirmed.

4 Rhombic Inventory Vector, all (n,k)-Ovals for $n \leq 10$

We use $\subseteq_m$ to denote containment in multisets. For example, if multiset $M = \{1,1,1,2,3,3,4,4,4,4\}$ then $L = \{1,1,1,2,4,4\} \subseteq_m M$ but $L' = \{1,1,1,2,2\} \not\subseteq_m M$. We say that L is a multisubset of M. Further, we replace $\underbrace{a,a,\ldots,a}_{b}$ by a^b, so $M = \{1^3, 2^1, 3^2, 4^4\}$.

On p.141 of Ball and Coxeter [1] it is proved that every (n,k)-Oval $\mathcal{O}$, with $2 \leq k \leq n$, can be tiled by a multiset of $\binom{k}{2}$ rhombs chosen from $\rho_1, \rho_2, \ldots, \rho_{\lfloor \frac{n}{2} \rfloor}$.

The regular $2n$-gon, $\{2n\}$, is an (n,n)-Oval with TAIS=$\underbrace{[1\ 1\ \cdots\ 1]}_{n}$.

Definition 4.1 The *Standard Rhombic Inventory*, SRI_{2n}, is the multiset of $\binom{n}{2}$ rhombs that tile $\{2n\}$.

There are $\lfloor \frac{n}{2} \rfloor$ different shapes of rhombs in SRI_{2n}; see Section 2. When n is odd, SRI_{2n} contains n copies of each of the $\frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor$ shapes of rhomb,

$\rho_1, \rho_2, \ldots, \rho_{\frac{n-1}{2}}$. When n is even, SRI_{2n} contains n copies of each of the $\frac{n}{2}-1$ non-square rhombs, $\rho_1, \rho_2, \ldots, \rho_{\frac{n}{2}-1}$, but only $\frac{n}{2}$ copies of the square $\rho_{\frac{n}{2}}$.

For a fixed (n,k)-Oval $\mathcal{O}$ let λ_h equal the number of rhombs in $\mathcal{O}$ with principal index h.

Definition 4.2 The *Rhombic Inventory Vector* (RIV) of Oval $\mathcal{O}$, RIV($\mathcal{O}$), is the vector $(\lambda_1, \lambda_2, \ldots, \lambda_{\lfloor \frac{n}{2} \rfloor})$ of length $\lfloor \frac{n}{2} \rfloor$.

The sum of the components in RIV($\mathcal{O}$) equals $\binom{k}{2}$.

Example 4.3 $(n,k) = (15,6)$. See Figs. 1 and 3. The $(15,6)$-Oval $\mathcal{X}$ is tiled by $\binom{6}{2} = 15$ rhombs. The rhomb ρ_4 occurs twice in $\mathcal{X}$, so $\lambda_4 = 2$. We have $\mathrm{RIV}(\mathcal{X}) = (2,1,2,2,4,2,2)$.

The RIV of an (n,k)-Oval can be derived from its TAIS by constructing its Oval Index Triangle, (OIT). The construction of an OIT is described below for our $(15,6)$-Oval $\mathcal{X}$.

First we define the function $r : \mathbb{Z}_n \backslash \{0\} \mapsto \mathbb{Z}_n \backslash \{0\}$:

$$r(a) = \begin{cases} a & \text{if } a \leq \lfloor \frac{n}{2} \rfloor, \\ -a \text{ or } n-a & \text{if } a > \lfloor \frac{n}{2} \rfloor. \end{cases} \tag{2}$$

We extend the definition of r to multisets M as follows: $r(M) = \{r(a) \mid a \in M\}$.

The TAIS for $\mathcal{X}$ is [4 3 2 1 4 1]. To compute RIV($\mathcal{X}$):

(i) Delete the last turning angle index from the TAIS, thereby obtaining the sequence of indices for the upper interior face angles of the rhombs in the *receptacle* — the cluster of $k-1$ rhombs that are incident on the stem vertex of the Oval. ('Receptacle' is the term used by botanists to denote the part of a plant that holds the fruit.) We call this sequence the 'truncated TAIS'. The truncated TAIS for $\mathcal{X}$ is [4 3 2 1 4].

(ii) The first row of the OIT equals the truncated TAIS. Below each pair of consecutive indices in the first row enter their sum in the second row:

$$\begin{array}{ccccccccc} 4 & & 3 & & 2 & & 1 & & 4 \\ & 7 & & 5 & & 3 & & 5 & \end{array}$$

(iii) Let $h_{i,j}$ denote the index in row i and position j of the triangle, where $i \geq 3$, and $j = 1, 2, \ldots, k-i$, counting from the left. Now the indices at each interior vertex of an (n,k)-Oval sum to $2n$, so simple trigonometry gives:

$$h_{i+1,j} = h_{i,j} + h_{i,j+1} - h_{i-1,j+1}.$$

See the left-hand triangle in (iv) below.

(iv) Apply function r to the indices of the left-hand triangle, *i.e.*, replace index $h > \lfloor \frac{n}{2} \rfloor$ by $n - h$. The OIT is now complete.

$$\begin{array}{ccccccccc} 4 & & 3 & & 2 & & 1 & & 4 \\ & 7 & & 5 & & 3 & & 5 & \\ & & 9 & & 6 & & 7 & & \\ & & & 10 & & 10 & & & \\ & & & & 14 & & & & \end{array} \quad \xrightarrow{r} \quad \begin{array}{ccccccccc} 4 & & 3 & & 2 & & 1 & & 4 \\ & 7 & & 5 & & 3 & & 5 & \\ & & 6 & & 6 & & 7 & & \\ & & & 5 & & 5 & & & \\ & & & & 1 & & & & \\ & & & & \text{OIT} & & & & \end{array}$$

(v) Now count the frequency of each principal index in the OIT to obtain $\mathrm{RIV}(\mathcal{X}) = (2, 1, 2, 2, 4, 2, 2)$, as above.

Recall the definition of $\alpha(S)$ from Definitions 2.8(1).

Definitions 4.4 $\delta(S)$, $\mathrm{OIT}(\alpha(S))$ or $\mathrm{OIT}(T)$

Let $S = \{s_1, s_2, \ldots, s_k\} \subseteq \mathbb{Z}_n$.

(1) $\delta(S) = \{s_j - s_i : 1 \leq i < j \leq k\}$ is a multiset of non-zero differences of S. Note that $|\delta(S)| = \binom{k}{2}$.

(2) $\mathrm{OIT}(\alpha(S)) = \mathrm{OIT}(T)$ is the multiset of indices in the OIT with first row $[s_2 - s_1, s_3 - s_2, \ldots, s_k - s_{k-1}]$, the truncation of $\alpha(S) = T$.

Lemma 4.5 *Let $S = \{s_1, s_2, \ldots, s_k\} \subseteq \mathbb{Z}_n$. Then $OIT(\alpha(S)) = r(\delta(S))$.*

Proof. Consider the triangle formed previously with $h_{i,j}$ as the index in row i and position j, counting from the left, and let H denote the multiset of all such $h_{i,j}$.

We show for $i = 1, 2, \ldots, k-1$, and $j = 1, 2, \ldots, k-i$ that $h_{i,j} = s_{i+j} - s_j \in \delta(S)$, *i.e.*, that the indices in row i of this triangle are the difference of two s's $\in S$ whose subscripts differ by i.

By definition of the triangle this is clearly true for $i = 1, 2$. Assume that the hypothesis is true for rows $1, 2, \ldots, i$. Then, for $i \geq 3$:

$$\begin{aligned} h_{i+1,j} &= h_{i,j} + h_{i,j+1} - h_{i-1,j+1} \\ &= (s_{i+j} - s_j) + (s_{i+(j+1)} - s_{j+1}) - (s_{(i-1)+(j+1)} - s_{j+1}) \\ &= s_{(i+1)+j} - s_j \in \delta(S), \end{aligned}$$

using strong induction at the second line. Hence the induction goes through, and $H \subseteq_{\mathrm{m}} \delta(S)$, but $|H| = \binom{k}{2} = |\delta(S)|$, and so $H = \delta(S)$. Now apply r to both sides of this equation to give the result. □

Example 4.6 $(n, k) = (15, 6)$. Our $(15, 6)$-Oval $\mathcal{X}$ has TAIS $T = [4\ 3\ 2\ 1\ 4\ 1]$. So $X = \beta(T) = \{0, 4, 7, 9, 10, 14\}$, giving $\delta(X) = \{1^1, 2^1, 3^2, 4^2, 5^2, 6^1, 7^2, 9^1, 10^2, 14^1\}$, and $r(\delta(X)) = \{1^2, 2^1, 3^2, 4^2, 5^4, 6^2, 7^2\}$. So $\mathrm{RIV}(\mathcal{X}) = (2, 1, 2, 2, 4, 2, 2)$, as above.

Remark 4.7 It is straightforward to show that the multiset $\mathrm{OIT}(T)$ doesn't depend on how we truncated T to form the first row of the OIT.

4.1 All (n, k)-Ovals and their RIV's for $n \leq 10$

In Tables 2 and 3 below we list and number all (n, k)-Ovals up to congruence, and their RIV's, for $2 \leq n \leq 10$. We refer to these Ovals by their numbers in later Sections.

$\mathcal{O}_i$	k	TAIS	RIV
$\mathcal{O}_1$	2	[1 1]	(1)

$n = 2$

$\mathcal{O}_i$	k	TAIS	RIV
$\mathcal{O}_1$	2	[1 2]	(1)
$\mathcal{O}_2$	3	[1 1 1]	(3)

$n = 3$

$\mathcal{O}_i$	k	TAIS	RIV
$\mathcal{O}_1$	2	[1 3]	(1, 0)
$\mathcal{O}_2$	2	[2 2]	(0, 1)
$\mathcal{O}_3$	3	[1 1 2]	(2, 1)
$\mathcal{O}_4$	4	[1 1 1 1]	(4, 2)

$n = 4$

$\mathcal{O}_i$	k	TAIS	RIV
$\mathcal{O}_1$	2	[1 4]	(1, 0)
$\mathcal{O}_2$	2	[2 3]	(0, 1)
$\mathcal{O}_3$	3	[1 1 3]	(2, 1)
$\mathcal{O}_4$	3	[1 2 2]	(1, 2)
$\mathcal{O}_5$	4	[1 1 1 2]	(3, 3)
$\mathcal{O}_6$	5	[1 1 1 1 1]	(5, 5)

$n = 5$

$\mathcal{O}_i$	k	TAIS	RIV
$\mathcal{O}_1$	2	[1 5]	(1, 0, 0)
$\mathcal{O}_2$	2	[2 4]	(0, 1, 0)
$\mathcal{O}_3$	2	[3 3]	(0, 0, 1)
$\mathcal{O}_4$	3	[1 1 4]	(2, 1, 0)
$\mathcal{O}_5$	3	[1 2 3]	(1, 1, 1)
$\mathcal{O}_6$	3	[2 2 2]	(0, 3, 0)
$\mathcal{O}_7$	4	[1 1 1 3]	(3, 2, 1)
$\mathcal{O}_8$	4	[1 1 2 2]	(2, 3, 1)
$\mathcal{O}_9$	4	[1 2 1 2]	(2, 2, 2)
$\mathcal{O}_{10}$	5	[1 1 1 1 2]	(4, 4, 2)
$\mathcal{O}_{11}$	6	[1 1 1 1 1 1]	(6, 6, 3)

$n = 6$

$\mathcal{O}_i$	k	TAIS	RIV
$\mathcal{O}_1$	2	[1 6]	(1, 0, 0)
$\mathcal{O}_2$	2	[2 5]	(0, 1, 0)
$\mathcal{O}_3$	2	[3 4]	(0, 0, 1)
$\mathcal{O}_4$	3	[1 1 5]	(2, 1, 0)
$\mathcal{O}_5$	3	[1 2 4]	(1, 1, 1)
$\mathcal{O}_6$	3	[1 3 3]	(1, 0, 2)
$\mathcal{O}_7$	3	[2 2 3]	(0, 2, 1)
$\mathcal{O}_8$	4	[1 1 1 4]	(3, 2, 1)
$\mathcal{O}_9$	4	[1 1 2 3]	(2, 2, 2)
$\mathcal{O}_{10}$	4	[1 2 1 3]	(2, 1, 3)
$\mathcal{O}_{11}$	4	[1 2 2 2]	(1, 3, 2)
$\mathcal{O}_{12}$	5	[1 1 1 1 3]	(4, 3, 3)
$\mathcal{O}_{13}$	5	[1 1 1 2 2]	(3, 4, 3)
$\mathcal{O}_{14}$	5	[1 1 2 1 2]	(3, 3, 4)
$\mathcal{O}_{15}$	6	[1 1 1 1 1 2]	(5, 5, 5)
$\mathcal{O}_{16}$	7	[1 1 1 1 1 1 1]	(7, 7, 7)

$n = 7$

Table 2: All (n, k)-Ovals up to congruence and their RIV's for $2 \leq n \leq 7$.

5 Magic Ovals, cyclic difference sets, multiplier -1, all magic (n, k, λ)-Ovals for $n \leq 40$

Recall $S = \{s_1, s_2, \ldots, s_k\} \subseteq \mathbb{Z}_n$, and $r : \mathbb{Z}_n \backslash \{0\} \mapsto \mathbb{Z}_n \backslash \{0\}$ from Equation (2), and $\delta(S)$ from Definitions 4.4(1); let M be a multiset with elements from $\mathbb{Z}_n \backslash \{0\}$. We need two more definitions.

Definitions 5.1 $f_M(a)$, $\Delta(S)$

(1) $f_M(a)$ is the frequency of $a \in M$.

(2) $\Delta(S) = \delta(S) \cup -\delta(S)$ is the multiset of non-zero differences of S.

$\mathcal{O}_i$	k	TAIS	RIV
$\mathcal{O}_1$	2	[1 7]	(1, 0, 0, 0)
$\mathcal{O}_2$	2	[2 6]	(0, 1, 0, 0)
$\mathcal{O}_3$	2	[3 5]	(0, 0, 1, 0)
$\mathcal{O}_4$	2	[4 4]	(0, 0, 0, 1)
$\mathcal{O}_5$	3	[1 1 6]	(2, 1, 0, 0)
$\mathcal{O}_6$	3	[1 2 5]	(1, 1, 1, 0)
$\mathcal{O}_7$	3	[1 3 4]	(1, 0, 1, 1)
$\mathcal{O}_8$	3	[2 2 4]	(0, 2, 0, 1)
$\mathcal{O}_9$	3	[2 3 3]	(0, 1, 2, 0)
$\mathcal{O}_{10}$	4	[1 1 1 5]	(3, 2, 1, 0)
$\mathcal{O}_{11}$	4	[1 1 2 4]	(2, 2, 1, 1)
$\mathcal{O}_{12}$	4	[1 1 3 3]	(2, 1, 2, 1)
$\mathcal{O}_{13}$	4	[1 2 1 4]	(2, 1, 2, 1)
$\mathcal{O}_{14}$	4	[1 2 2 3]	(1, 2, 2, 1)
$\mathcal{O}_{15}$	4	[1 2 3 2]	(1, 2, 3, 0)
$\mathcal{O}_{16}$	4	[1 3 1 3]	(2, 0, 2, 2)
$\mathcal{O}_{17}$	4	[2 2 2 2]	(0, 4, 0, 2)
$\mathcal{O}_{18}$	5	[1 1 1 1 4]	(4, 3, 2, 1)
$\mathcal{O}_{19}$	5	[1 1 1 2 3]	(3, 3, 3, 1)
$\mathcal{O}_{20}$	5	[1 1 2 1 3]	(3, 2, 3, 2)
$\mathcal{O}_{21}$	5	[1 1 2 2 2]	(2, 4, 2, 2)
$\mathcal{O}_{22}$	5	[1 2 1 2 2]	(2, 3, 4, 1)
$\mathcal{O}_{23}$	6	[1 1 1 1 1 3]	(5, 4, 4, 2)
$\mathcal{O}_{24}$	6	[1 1 1 1 2 2]	(4, 5, 4, 2)
$\mathcal{O}_{25}$	6	[1 1 1 2 1 2]	(4, 4, 5, 2)
$\mathcal{O}_{26}$	6	[1 1 2 1 1 2]	(4, 4, 4, 3)
$\mathcal{O}_{27}$	7	[1 1 1 1 1 1 2]	(6, 6, 6, 3)
$\mathcal{O}_{28}$	8	[1 1 1 1 1 1 1]	(8, 8, 8, 4)

$n = 8$

$\mathcal{O}_i$	k	TAIS	RIV
$\mathcal{O}_1$	2	[1 8]	(1, 0, 0, 0)
$\mathcal{O}_2$	2	[2 7]	(0, 1, 0, 0)
$\mathcal{O}_3$	2	[3 6]	(0, 0, 1, 0)
$\mathcal{O}_4$	2	[4 5]	(0, 0, 0, 1)
$\mathcal{O}_5$	3	[1 1 7]	(2, 1, 0, 0)
$\mathcal{O}_6$	3	[1 2 6]	(1, 1, 1, 0)
$\mathcal{O}_7$	3	[1 3 5]	(1, 0, 1, 1)
$\mathcal{O}_8$	3	[1 4 4]	(1, 0, 0, 2)
$\mathcal{O}_9$	3	[2 2 5]	(0, 2, 0, 1)
$\mathcal{O}_{10}$	3	[2 3 4]	(0, 1, 1, 1)
$\mathcal{O}_{11}$	3	[3 3 3]	(0, 0, 3, 0)
$\mathcal{O}_{12}$	4	[1 1 1 6]	(3, 2, 1, 0)
$\mathcal{O}_{13}$	4	[1 1 2 5]	(2, 2, 1, 1)
$\mathcal{O}_{14}$	4	[1 1 3 4]	(2, 1, 1, 2)
$\mathcal{O}_{15}$	4	[1 2 1 5]	(2, 1, 2, 1)
$\mathcal{O}_{16}$	4	[1 2 2 4]	(1, 2, 1, 2)
$\mathcal{O}_{17}$	4	[1 2 3 3]	(1, 1, 3, 1)
$\mathcal{O}_{18}$	4	[1 2 4 2]	(1, 2, 2, 1)
$\mathcal{O}_{19}$	4	[1 3 1 4]	(2, 0, 1, 3)
$\mathcal{O}_{20}$	4	[1 3 2 3]	(1, 1, 2, 2)
$\mathcal{O}_{21}$	4	[2 2 2 3]	(0, 3, 1, 2)
$\mathcal{O}_{22}$	5	[1 1 1 1 5]	(4, 3, 2, 1)
$\mathcal{O}_{23}$	5	[1 1 1 2 4]	(3, 3, 2, 2)
$\mathcal{O}_{24}$	5	[1 1 1 3 3]	(3, 2, 3, 2)
$\mathcal{O}_{25}$	5	[1 1 2 1 4]	(3, 2, 2, 3)
$\mathcal{O}_{26}$	5	[1 1 2 2 3]	(2, 3, 2, 3)
$\mathcal{O}_{27}$	5	[1 1 2 3 2]	(2, 3, 3, 2)
$\mathcal{O}_{28}$	5	[1 1 3 1 3]	(3, 1, 2, 4)
$\mathcal{O}_{29}$	5	[1 2 1 2 3]	(2, 2, 4, 2)
$\mathcal{O}_{30}$	5	[1 2 2 1 3]	(2, 2, 3, 3)
$\mathcal{O}_{31}$	5	[1 2 2 2 2]	(1, 4, 2, 3)
$\mathcal{O}_{32}$	6	[1 1 1 1 1 4]	(5, 4, 3, 3)
$\mathcal{O}_{33}$	6	[1 1 1 1 2 3]	(4, 4, 4, 3)
$\mathcal{O}_{34}$	6	[1 1 1 2 1 3]	(4, 3, 4, 4)
$\mathcal{O}_{35}$	6	[1 1 1 2 2 2]	(3, 5, 3, 4)
$\mathcal{O}_{36}$	6	[1 1 2 1 1 3]	(4, 3, 3, 5)
$\mathcal{O}_{37}$	6	[1 1 2 1 2 2]	(3, 4, 4, 4)
$\mathcal{O}_{38}$	6	[1 2 1 2 1 2]	(3, 3, 6, 3)
$\mathcal{O}_{39}$	7	[1 1 1 1 1 1 3]	(6, 5, 5, 5)
$\mathcal{O}_{40}$	7	[1 1 1 1 1 2 2]	(5, 6, 5, 5)
$\mathcal{O}_{41}$	7	[1 1 1 1 2 1 2]	(5, 5, 6, 5)
$\mathcal{O}_{42}$	7	[1 1 1 2 1 1 2]	(5, 5, 5, 6)
$\mathcal{O}_{43}$	8	[1 1 1 1 1 1 1 2]	(7, 7, 7, 7)
$\mathcal{O}_{44}$	9	[1 1 1 1 1 1 1 1 1]	(9, 9, 9, 9)

$n = 9$

$\mathcal{O}_i$	k	TAIS	RIV
$\mathcal{O}_1$	2	[1 9]	(1, 0, 0, 0, 0)
$\mathcal{O}_2$	2	[2 8]	(0, 1, 0, 0, 0)
$\mathcal{O}_3$	2	[3 7]	(0, 0, 1, 0, 0)
$\mathcal{O}_4$	2	[4 6]	(0, 0, 0, 1, 0)
$\mathcal{O}_5$	2	[5 5]	(0, 0, 0, 0, 1)
$\mathcal{O}_6$	3	[1 1 8]	(2, 1, 0, 0, 0)
$\mathcal{O}_7$	3	[1 2 7]	(1, 1, 1, 0, 0)
$\mathcal{O}_8$	3	[1 3 6]	(1, 0, 1, 1, 0)
$\mathcal{O}_9$	3	[1 4 5]	(1, 0, 0, 1, 1)
$\mathcal{O}_{10}$	3	[2 2 6]	(0, 2, 0, 1, 0)
$\mathcal{O}_{11}$	3	[2 3 5]	(0, 1, 1, 0, 1)
$\mathcal{O}_{12}$	3	[2 4 4]	(0, 1, 0, 2, 0)
$\mathcal{O}_{13}$	3	[3 3 4]	(0, 0, 2, 1, 0)
$\mathcal{O}_{14}$	4	[1 1 1 7]	(3, 2, 1, 0, 0)
$\mathcal{O}_{15}$	4	[1 1 2 6]	(2, 2, 1, 1, 0)
$\mathcal{O}_{16}$	4	[1 1 3 5]	(2, 1, 1, 1, 1)
$\mathcal{O}_{17}$	4	[1 1 4 4]	(2, 1, 0, 2, 1)
$\mathcal{O}_{18}$	4	[1 2 1 6]	(2, 1, 2, 1, 0)
$\mathcal{O}_{19}$	4	[1 2 2 5]	(1, 2, 1, 1, 1)
$\mathcal{O}_{20}$	4	[1 2 3 4]	(1, 1, 2, 1, 1)
$\mathcal{O}_{21}$	4	[1 2 4 3]	(1, 1, 2, 2, 0)
$\mathcal{O}_{22}$	4	[1 2 5 2]	(1, 2, 2, 0, 1)
$\mathcal{O}_{23}$	4	[1 3 1 5]	(2, 0, 1, 2, 1)
$\mathcal{O}_{24}$	4	[1 3 2 4]	(1, 1, 1, 2, 1)
$\mathcal{O}_{25}$	4	[1 3 3 3]	(1, 0, 3, 2, 0)
$\mathcal{O}_{26}$	4	[1 4 1 4]	(2, 0, 0, 2, 2)
$\mathcal{O}_{27}$	4	[2 2 2 4]	(0, 3, 0, 3, 0)
$\mathcal{O}_{28}$	4	[2 2 3 3]	(0, 2, 2, 1, 1)
$\mathcal{O}_{29}$	4	[2 3 2 3]	(0, 2, 2, 0, 2)
$\mathcal{O}_{30}$	5	[1 1 1 1 6]	(4, 3, 2, 1, 0)
$\mathcal{O}_{31}$	5	[1 1 1 2 5]	(3, 3, 2, 1, 1)
$\mathcal{O}_{32}$	5	[1 1 1 3 4]	(3, 2, 2, 2, 1)
$\mathcal{O}_{33}$	5	[1 1 2 1 5]	(3, 2, 2, 2, 1)
$\mathcal{O}_{34}$	5	[1 1 2 2 4]	(2, 3, 1, 3, 1)
$\mathcal{O}_{35}$	5	[1 1 2 3 3]	(2, 2, 3, 2, 1)
$\mathcal{O}_{36}$	5	[1 1 2 4 2]	(2, 3, 2, 3, 0)
$\mathcal{O}_{37}$	5	[1 1 3 1 4]	(3, 1, 1, 3, 2)
$\mathcal{O}_{38}$	5	[1 1 3 2 3]	(2, 2, 2, 2, 2)
$\mathcal{O}_{39}$	5	[1 2 1 2 4]	(2, 2, 3, 2, 1)
$\mathcal{O}_{40}$	5	[1 2 1 3 3]	(2, 1, 4, 3, 0)
$\mathcal{O}_{41}$	5	[1 2 2 2 3]	(1, 3, 2, 3, 1)
$\mathcal{O}_{42}$	5	[1 2 2 3 2]	(1, 3, 3, 1, 2)
$\mathcal{O}_{43}$	5	[1 2 2 1 4]	(2, 2, 2, 2, 2)
$\mathcal{O}_{44}$	5	[1 2 3 1 3]	(2, 1, 3, 3, 1)
$\mathcal{O}_{45}$	5	[2 2 2 2 2]	(0, 5, 0, 5, 0)
$\mathcal{O}_{46}$	6	[1 1 1 1 1 5]	(5, 4, 3, 2, 1)
$\mathcal{O}_{47}$	6	[1 1 1 1 2 4]	(4, 4, 3, 3, 1)
$\mathcal{O}_{48}$	6	[1 1 1 1 3 3]	(4, 3, 4, 3, 1)
$\mathcal{O}_{49}$	6	[1 1 1 2 1 4]	(4, 3, 3, 3, 2)
$\mathcal{O}_{50}$	6	[1 1 1 2 2 3]	(3, 4, 3, 3, 2)
$\mathcal{O}_{51}$	6	[1 1 1 2 3 2]	(3, 4, 4, 2, 2)
$\mathcal{O}_{52}$	6	[1 1 1 3 1 3]	(4, 2, 3, 4, 2)
$\mathcal{O}_{53}$	6	[1 1 2 1 1 4]	(4, 3, 2, 4, 2)
$\mathcal{O}_{54}$	6	[1 1 2 1 2 3]	(3, 3, 4, 3, 2)
$\mathcal{O}_{55}$	6	[1 1 2 1 3 2]	(3, 3, 4, 4, 1)
$\mathcal{O}_{56}$	6	[1 1 2 2 1 3]	(3, 3, 3, 4, 2)
$\mathcal{O}_{57}$	6	[1 1 2 2 2 2]	(2, 5, 2, 5, 1)
$\mathcal{O}_{58}$	6	[1 1 3 1 1 3]	(4, 2, 2, 4, 3)
$\mathcal{O}_{59}$	6	[1 2 1 2 1 3]	(3, 2, 5, 4, 1)
$\mathcal{O}_{60}$	6	[1 2 1 2 2 2]	(2, 4, 4, 3, 2)
$\mathcal{O}_{61}$	6	[1 2 2 1 2 2]	(2, 4, 4, 2, 3)
$\mathcal{O}_{62}$	7	[1 1 1 1 1 1 4]	(6, 5, 4, 4, 2)
$\mathcal{O}_{63}$	7	[1 1 1 1 1 2 3]	(5, 5, 5, 4, 2)
$\mathcal{O}_{64}$	7	[1 1 1 1 2 1 3]	(5, 4, 5, 5, 2)
$\mathcal{O}_{65}$	7	[1 1 1 1 2 2 2]	(4, 6, 4, 5, 2)
$\mathcal{O}_{66}$	7	[1 1 1 2 1 1 3]	(5, 4, 4, 5, 3)
$\mathcal{O}_{67}$	7	[1 1 1 2 1 2 2]	(4, 5, 5, 4, 3)
$\mathcal{O}_{68}$	7	[1 1 2 1 1 2 2]	(4, 5, 4, 6, 2)
$\mathcal{O}_{69}$	7	[1 1 2 1 2 1 2]	(4, 4, 6, 5, 2)
$\mathcal{O}_{70}$	8	[1 1 1 1 1 1 1 3]	(7, 6, 6, 6, 3)
$\mathcal{O}_{71}$	8	[1 1 1 1 1 1 2 2]	(6, 7, 6, 6, 3)
$\mathcal{O}_{72}$	8	[1 1 1 1 1 2 1 2]	(6, 6, 7, 6, 3)
$\mathcal{O}_{73}$	8	[1 1 1 1 2 1 1 2]	(6, 6, 6, 7, 3)
$\mathcal{O}_{74}$	8	[1 1 1 2 1 1 1 2]	(6, 6, 6, 6, 4)
$\mathcal{O}_{75}$	9	[1 1 1 1 1 1 1 1 2]	(8, 8, 8, 8, 4)
$\mathcal{O}_{76}$	10	[1 1 1 1 1 1 1 1 1 1]	(10, 10, 10, 10, 5)

$n = 10$

Table 3: All (n, k)-Ovals up to congruence and their RIV's for $8 \leq n \leq 10$.

Note that $-\delta(S) = \{s_i - s_j : 1 \leq i < j \leq k\}$, and $|-\delta(S)| = |\delta(S)| = \binom{k}{2}$, and $|\Delta(S)| = k(k-1)$.

Lemma 5.2 *Let M be a multiset with elements from $\mathbb{Z}_n \backslash \{0\}$. Then $r(M) = r(-M)$.*

Proof. Let n be even. Consider an occurrence of $a \in M$.

Suppose $a \leq \lfloor \frac{n}{2} \rfloor$. First, if $a = \frac{n}{2}$ then $r(a) = \frac{n}{2}$. Now $-a = \frac{n}{2} \in -M$ and $r(-a) = \frac{n}{2}$ also. Thus element $\frac{n}{2} \in M$ 'contributes' the same element $\frac{n}{2}$ to both multisets $r(M)$ and $r(-M)$. Second, if $a < \lfloor \frac{n}{2} \rfloor$ then $r(a) = a$. Now $-a \in -M$ satisfies $-a > \lfloor \frac{n}{2} \rfloor$ so $r(-a) = -(-a) = a$. So, again, element $a \in M$ contributes the same element a to both $r(M)$ and $r(-M)$.

Suppose $a > \lfloor \frac{n}{2} \rfloor$. Then $r(a) = -a$. Now $-a \in -M$ satisfies $-a < \lfloor \frac{n}{2} \rfloor \leq \lfloor \frac{n}{2} \rfloor$ so $r(-a) = -a$. Thus, element $a \in M$ contributes the same element $-a$ to both $r(M)$ and $r(-M)$.

In conclusion, any occurrence of $a \in M$ contributes the same element to both multisets $r(M)$ and $r(-M)$. Thus $r(M) = r(-M)$. The proof for odd n is similar. □

Definition 5.3 The *Short Frequency Vector* (SFV) of $r(M)$ is the vector $(f_{r(M)}(1), f_{r(M)}(2), \ldots, f_{r(M)}(\lfloor \frac{n}{2} \rfloor))$ of length $\lfloor \frac{n}{2} \rfloor$.

Remark 5.4 From Lemma 4.5 we have $\mathrm{RIV}(\mathcal{O}(\alpha(S))) = \mathrm{SFV}(r(\delta(S)))$.

Example 5.5 $(n, k) = (15, 6)$. See Example 4.6. Here $X = \{0, 4, 7, 9, 10, 14\} \subseteq \mathbb{Z}_{15}$ and $\delta(X) = \{1^1, 2^1, 3^2, 4^2, 5^2, 6^1, 7^2, 9^1, 10^2, 14^1\}$, and $r(\delta(X)) = \{1^2, 2^1, 3^2, 4^2, 5^4, 6^2, 7^2\}$. So $\mathrm{RIV}(\mathcal{O}(\alpha(X))) = \mathrm{SFV}(r(\delta(X))) = (2, 1, 2, 2, 4, 2, 2)$.

Lemma 5.6 *Let $S \subseteq \mathbb{Z}_n$. Then $SFV(r(\Delta(S))) = 2 \times SFV(r(\delta(S)))$.*

Proof. Now $\Delta(S) = \delta(S) \cup -\delta(S)$, and so $r(\Delta(S)) = r(\delta(S)) \cup -r(\delta(S)) = r(\delta(S)) \cup r(\delta(S))$ using Lemma 5.2. Hence for any $a \in r(\delta(S))$ we have $f_{r(\Delta(S))}(a) = 2 \times f_{r(\delta(S))}(a)$, and so the result. □

Example 5.7 $(n, k) = (15, 6)$. See Example 5.5. Again, $X = \{0, 4, 7, 9, 10, 14\} \subseteq \mathbb{Z}_{15}$ and $\Delta(X) = \{1^2, 2^2, 3^4, 4^4, 5^4, 6^2, 7^4, 9^2, 10^4, 14^2\}$, and $r(\Delta(X)) = \{1^4, 2^2, 3^4, 4^4, 5^8, 6^4, 7^4\}$. So $\mathrm{SFV}(r(\Delta(X))) = (4, 2, 4, 4, 8, 4, 4) = 2 \times (2, 1, 2, 2, 4, 2, 2) = 2\times\ \mathrm{SFV}(r(\delta(X)))$.

5.1 Magic Ovals and cyclic difference sets

Definition 5.8 A (n,k,λ)-*cyclic difference set* – (n,k,λ)-CDS – is a k-subset $D \subseteq \mathbb{Z}_n$ with the property that $\Delta(D)$ contains every non-zero element of $\mathbb{Z}_n$ exactly λ times.

In a (n,k,λ)-CDS straightforward counting gives:

$$\lambda(n-1) = k(k-1), \tag{3}$$

this shows that λ is even if n is even.

Example 5.9 $(n,k) = (7,3)$. $D = \{0,1,3\}$ is a $(7,3,1)$-CDS. We have $\delta(D) = \{1,3,2\}$ and $-\delta(D) = \{-1,-3,-2\} = \{6,4,5\}$, giving $\Delta(D) = \{1^1, 2^1, 3^1, 4^1, 5^1, 6^1\}$.

Recall that, when n is odd, there are n copies of each of the $\lfloor \frac{n}{2} \rfloor$ distinct rhombs in SRI_{2n}, *i.e.*, $\mathrm{RIV}(\{2n\}) = (n,n,\ldots,n,n)$, and, when n is even, there are n copies of each of the $\frac{n}{2}-1$ non-square rhombs in SRI_{2n}, but only $\frac{n}{2}$ copies of the square, *i.e.*, $\mathrm{RIV}(\{2n\}) = (n,n,\ldots,n,\frac{n}{2})$.

Definition 5.10 A *magic* (n,k,λ)-Oval is, for odd n, an (n,k)-Oval that contains exactly λ copies of each of the $\lfloor \frac{n}{2} \rfloor$ distinct rhombs of SRI_{2n}, *i.e.*, that has RIV$= (\lambda,\lambda,\ldots,\lambda,\lambda)$, and is, for even n, an (n,k)-Oval that contains exactly λ copies of each of the $\frac{n}{2}-1$ non-square rhombs in SRI_{2n}, but only $\frac{\lambda}{2}$ copies of the square, *i.e.*, that has RIV$= (\lambda,\lambda,\ldots,\lambda,\frac{\lambda}{2})$.

The following Theorem 5.11 is a main result, it proves equivalence of a magic (n,k,λ)-Oval and a (n,k,λ)-CDS.

Theorem 5.11 *Let $S = \{s_1, s_2, \ldots, s_k\} \subseteq \mathbb{Z}_n$. Then $\mathcal{O}(\alpha(S))$ is a magic (n,k,λ)-Oval if and only if S is a (n,k,λ)-CDS. Moreover, λ is equal to the number of 1's in TAIS $\alpha(S)$.*

Proof. Necessity: let $\mathcal{O}(\alpha(S))$ be a magic (n,k,λ)-Oval.

For odd n: for each $h = 1,2,\ldots,\lfloor \frac{n}{2} \rfloor$, there are λ occurrences of h in $\mathrm{OIT}(\alpha(S))$ so, by the proof of Lemma 4.5, the multiset $\delta(S)$ contains λ occurrences from $\{h, n-h\}$. Suppose h occurs λ' times in $\delta(S)$ then $n-h$ will occur $\lambda-\lambda'$ times in $\delta(S)$, so h will occur $\lambda-\lambda'$ times in $-\delta(S)$. Hence h will occur exactly λ times in $\Delta(S) = \delta(S) \cup -\delta(S)$. For $h = \lfloor \frac{n}{2} \rfloor +$

$1, \lfloor \frac{n}{2} \rfloor + 2, \ldots, n-1$, we argue in a similar way with h replaced by $n-h$ to conclude that these h also occur λ times in $\Delta(S)$. Now $\Delta(S)$ is the multiset of differences defined by S; hence S is a cyclic difference set with repetition number λ, *i.e.*, S is a (n, k, λ)-CDS.

For even n: arguing as above each $h \neq \frac{n}{2}$ occurs λ times in $\Delta(S)$. Also $h = \frac{n}{2}$ occurs $\frac{\lambda}{2}$ times in $\mathrm{OIT}(\alpha(S))$, *i.e.*, $\frac{\lambda}{2}$ times in $r(\delta(S))$ and so $\frac{\lambda}{2}$ times in $\delta(S)$, and thus λ times in $\Delta(S)$ using Lemma 5.6. Hence, for even n also, S is a (n, k, λ)-CDS.

Sufficiency: let $S = \{s_1, s_2, \ldots, s_k\}$ be a (n, k, λ)-CDS. So, for odd n, we have $\mathrm{SFV}(r(\Delta(S))) = (2\lambda, 2\lambda, \ldots, 2\lambda, 2\lambda)$, and, for even n, we have $\mathrm{SFV}(r(\Delta(S))) = (2\lambda, 2\lambda, \ldots, 2\lambda, \lambda)$. Hence, from Lemma 5.6, for odd n, we have $\mathrm{SFV}(r(\delta(S))) = (\lambda, \lambda, \ldots, \lambda, \lambda)$, and, for even n, we have $\mathrm{SFV}(r(\delta(S))) = (\lambda, \lambda, \ldots, \lambda, \frac{\lambda}{2})$. But $\mathrm{RIV}(\mathcal{O}(\alpha(S))) = \mathrm{SFV}(r(\delta(S)))$ and so $\mathcal{O}(\alpha(S))$ is a magic (n, k, λ)-Oval.

Let μ be the number of 1's in TAIS $\alpha(S) = [s_2 - s_1, s_3 - s_2, \cdots, s_k - s_{k-1}, s_1 - s_k]$. Recall that the elements in $S = \{s_1, s_2, \ldots, s_k\}$ are in increasing order and satisfy $0 \leq s_1 < s_2 < \cdots < s_k$. There are λ 1's in $\Delta(S)$; hence there are λ solutions to $s_j - s_i \equiv 1 \pmod{n}$, where $i, j \in \{1, 2, \ldots, k\}$, $i \neq j$. Now if $s_j - s_i = 1$ or $-(n-1)$ then $j = i+1$ for $1 \leq i \leq k-1$, or $j = 1$ and $i = k$ (respectively), and thus $s_j - s_i$ is an element of $\alpha(S)$. Hence $\mu \geq \lambda$. Conversely, because there are μ 1's in the TAIS $\alpha(S)$ and every element of this TAIS is also an element of $\Delta(S)$, then $\mu \leq \lambda$. Hence $\lambda = \mu$. $\square$

Example 5.12

(a) The regular $2n$-gon $\{2n\}$ has TAIS= $\underbrace{[1\ 1\ \cdots\ 1]}_{n}$, which contains n 1's. It is a magic (n, n, n)-Oval with corresponding (n, n, n)-CDS $D = \{0, 1, \ldots, n-1\}$. For odd n we have $\mathrm{RIV}(\{2n\}) = (n, n, \ldots, n, n)$, and for even n $\mathrm{RIV}(\{2n\}) = (n, n, \ldots, n, \frac{n}{2})$.

(b) If we remove the right-hand strip of rhombs in $\{2n\}$ we produce a magic $(n, n-1, n-2)$-Oval $\{2n\}'$ with TAIS= $\underbrace{[1\ 1\ \cdots\ 1\ 2]}_{n-1}$, containing $n-2$ 1's. For odd n we have $\mathrm{RIV}(\{2n\}') = (n-2, n-2, \ldots, n-2, n-2)$, and, for even n, we have $\mathrm{RIV}(\{2n\}') = (n-2, n-2, \ldots, n-2, \frac{n-2}{2})$. The corresponding $(n, n-1, n-2)$-CDS is $D' = \{0, 1, \ldots, n-2\}$. See Fig. 5 for an example with $n = 12$.

If we remove another strip of rhombs we obtain an $(n, n-2)$-Oval but only

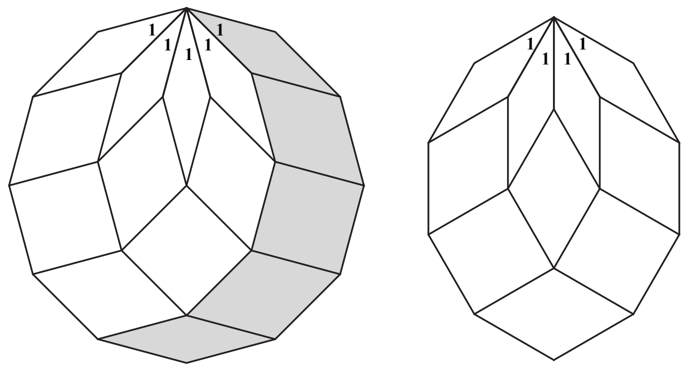


Figure 5: The regular 12-gon $\{12\}$, and the magic $(6,5,4)$-Oval $\{12\}'$ obtained by removing the right-hand strip of rhombs from $\{12\}$.

non-integer values of λ result from Equation (3), and so such an Oval is not magic.

(c) $(n,k) = (7,3)$. See Example 5.9. The set $D = \{0,1,3\}$ is a $(7,3,1)$-CDS, and so $\mathcal{O}(\alpha(D))$ is a magic $(7,3,1)$-Oval with TAIS $\alpha(D) = [1\ 2\ 4]$, which contains one 1. The OIT for $\mathcal{O}(\alpha(D))$ is $\begin{smallmatrix}1 & & 2\\ & 3 & \end{smallmatrix}$ and so $\mathrm{RIV}(\mathcal{O}(\alpha(D))) = (1,1,1)$. See the fourth $(7,3)$-Oval in Fig. 4.

(d) $(n,k) = (15,7)$. See Fig. 6. The set $D = \{0,1,2,4,5,8,10\}$ is a $(15,7,3)$-CDS. We have $\alpha(D) = [1\ 1\ 2\ 1\ 3\ 2\ 5]$, which contains 3 1's, and the $(15,7)$-Oval $\mathcal{O}(\alpha(D))$ is a magic $(15,7,3)$-Oval with OIT

$$\begin{array}{ccccccccccc}
1 & & 1 & & 2 & & 1 & & 3 & & 2\\
 & 2 & & 3 & & 3 & & 4 & & 5 & \\
 & & 4 & & 4 & & 6 & & 6 & & \\
 & & & 5 & & 7 & & 7 & & & \\
 & & & & 7 & & 6 & & & & \\
 & & & & & 5 & & & & &
\end{array} \qquad \text{and} \qquad \text{RIV } (3,3,3,3,3,3,3).$$

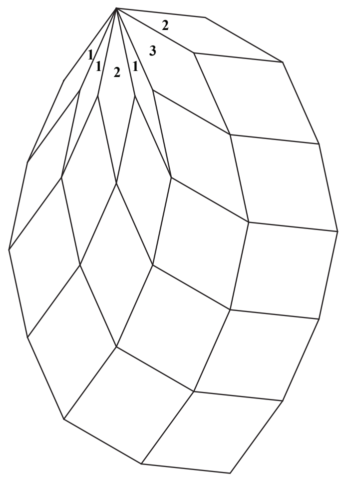


Figure 6: The magic $(15, 7, 3)$-Oval $\mathcal{O}([1\ 1\ 2\ 1\ 3\ 2\ 5])$.

Remark 5.13 The CDS's D and D' in Examples 5.12(a) and (b) above are usually considered to be 'trivial' CDS; see p.298 of [3]. We ignore the other two trivial CDS, namely $\emptyset$ and $\{s_i\}$, because $k \geq 2$. Thus non-trivial magic (n, k, λ)-Ovals have $2 \leq k \leq n-2$.

Both these trivial CDS's have $\mathrm{mult}(D) = \mathrm{mult}(D') = U(n)$, so both have -1 as a multiplier. Let D be a non-trivial (n, k, λ)-CDS. Then it is combinatorial folklore that -1 is *not* a multiplier of D; see the discussion on p.60 of Baumert [2]. Thus -1 is not a multiplier of the non-trivial magic (n, k, λ)-Oval $\mathcal{O}(\alpha(D))$. Then Theorem 3.7(ii) gives Theorem 5.14 below which is a geometrical interpretation of this fact.

Theorem 5.14 *Let $\mathcal{O}(\alpha(D))$ be a non-trivial magic (n, k, λ)-Oval. Then -1 is not a multiplier of $\mathcal{O}(\alpha(D))$, so $\mathcal{O}(\alpha(D)) \neq \mathcal{O}(\alpha(-D))$ and $\{\mathcal{O}(\alpha(D)), \mathcal{O}(\alpha(-D))\}$ is a congruent enantiomorphic pair in $\mathcal{O}^*(n, k)$.*

Example 5.15 $(n, k) = (7, 3)$. See Examples 3.8 and 5.12(c). The $(7, 3)$-Oval $\mathcal{O}(\alpha(\mathcal{D}))$ with $D = \{0, 1, 3\}$ is a non-trivial magic $(7, 3, 1)$-Oval, so $-1 \notin \mathrm{mult}(\mathcal{O}(\alpha(\mathcal{D})))$ and $\{\mathcal{O}(\alpha(D)), \mathcal{O}(\alpha(-D))\}$ is a congruent enantiomorphic pair in $\mathcal{O}^*(7, 3)$.

To the end of this Section we assume our CDS's are non-trivial.

Definition 5.16 A (n, k, λ)-CDS is *planar* if $\lambda = 1$.

We now give a new proof that -1 is not a multiplier of a planar CDS.

Theorem 5.17 *Let D be a planar $(n, k, 1)$-CDS with $k \geq 3$. Then $-1 \notin \mathrm{mult}(D)$.*

Proof. Let $T = \alpha(D) = [t_1\ t_2\ \cdots\ t_k]$ be the TAIS of $\mathcal{O}(\alpha(D))$. Then $\mathcal{O}(\alpha(D))$ is a magic $(n, k, 1)$-Oval. Suppose that two parts of T are equal, say $t_i = t_j = h$ for $1 \leq i < j \leq k$ and $1 \leq h \leq \lfloor \frac{n}{2} \rfloor$. Now form OIT$(T)$ using any truncated TAIS containing both t_i and t_j, this is possible because $k \geq 3$. Then OIT(T) will contain at least 2 copies of rhomb ρ_h, *i.e.*, $\lambda_h \geq 2$ in RIV$(\mathcal{O}(\alpha(D)))$, a contradiction because $\lambda = \lambda_h = 1$. So the k parts of $T = [t_1\ t_2\ \cdots\ t_k]$ are distinct.

Suppose that T is reversible, so $T \equiv_{\text{cyc}} \overleftarrow{T}$ where $\overleftarrow{T} = [t_k\ t_{k-1} \cdots\ t_1]$. Now, because the parts of T are distinct, we have $\overleftarrow{T} \equiv_{\text{cyc}} [t_1\ t_k \cdots\ t_2] = [t_1\ t_2\ \cdots\ t_k]$, so $t_k = t_2$, a contradiction. Hence T is not reversible, and, by Theorem 3.4, we have $-1 \notin \mathrm{mult}(D)$. □

5.2 All magic (n, k, λ)-Ovals, $n \leq 40$

See p.2 of Baumert [2].

Definition 5.18 Two k-subsets S and S' of $\mathbb{Z}_n$ are *(u,z)-equivalent*, $S \equiv_{u,z} S'$, if there exists $u \in U(n)$ and $z \in \mathbb{Z}_n$ such that $S = uS' + z$.

Table 6.1, p.150 of [2] contains a complete list of the 74 (n, k, λ) triples with $k \leq 100$ for which a (n, k, λ)-CDS exists, with at least one example of such a CDS for each triple.

Moreover, for the 12 (n, k, λ) triples with $n \leq 40$, see our Table 4 below, the (n, k, λ)-CDS examples in Table 6.1 of [2] are *all* the examples up to (u, z)-equivalence. To confirm this statement for these 12 triples see Hall [5]. As a double-check for the 8 triples: $(7, 3, 1)$, $(13, 4, 1)$, $(15, 7, 3)$, $(19, 9, 4)$, $(21, 5, 1)$, $(23, 11, 5)$, $(31, 6, 1)$, and $(37, 9, 2)$ see the explicit examples on pp.306–308 and p.327 of [3]. The remaining 4 triples: $(11, 5, 2)$, $(31, 15, 7)$, $(35, 17, 8)$, and $(40, 13, 4)$ were also double-checked by the authors using computer searches and Theorem 2.9 on p.306 of [3].

Amongst these 12 triples, for just one triple, namely $(31, 15, 7)$, there is more than one inequivalent (n, k, λ)-CDS: there are two inequivalent $(31, 15, 7)$-CDS's, these are labelled '31A' and '31B' in Table 6.1 of [2], and 'A' and 'B' in our Table 4.

We stopped at $n = 40$ in our Table 4 to indicate that magic (n, k, λ)-Ovals with n even can occur.

Remark 5.19 Now $-1 \notin \mathrm{mult}(D)$; hence $\mathrm{Mult}(D) = \mathrm{mult}(D) \cup - \mathrm{mult}(D)$ and $|\mathrm{Mult}(D)| = 2\,|\mathrm{mult}(D)|$ from Definition 3.11 and Remark 3.12.

Example 5.20 $(n, k) = (13, 4)$. The unique $(13, 4, 1)$-CDS up to (u, z)-equivalence is $D = \{0, 1, 3, 9\}$.

We have $\mathrm{mult}(D) = \{1, 3, 9\}$ and $\mathrm{Mult}(D) = \{1, 3, 4, 9, 10, 12\}$. Now $|U(13)| = 12$ so $|U(13) : \mathrm{Mult}(D)| = 2$. A set of 2 coset representatives for $\mathrm{Mult}(D)$ in $U(13)$ is $\{1, 2\}$. Then the 2 incongruent $(13, 4, 1)$-CDS's that are each (u, z)-equivalent to D are D and $2D = \{0, 2, 5, 6\} \equiv_z \{0, 1, 8, 10\}$, with corresponding TAIS's [1 2 6 4] and [1 3 2 7] respectively. Thus there are 2 magic $(13, 4, 1)$-Ovals up to congruency; see our Table 4.

A similar procedure applied to each (n, k, λ)-CDS of Table 6.1 of [2] for $n \leq 40$ produces our Table 4.

Example 5.21 $(n, k) = (16, 6)$. There does not exist a $(16, 6, 2)$-CDS; see Example 14.20(a) on p.425 of [3]. So there does not exist a magic $(16, 6, 2)$-Oval, *i.e.*, a $(16, 6)$-Oval with RIV $(2, 2, 2, 2, 2, 2, 2, 1)$. Consider the $(16, 6)$-Oval $\mathcal{O} = \mathcal{O}([1\ 1\ 2\ 1\ 5\ 6])$. Then $\mathrm{RIV}(\mathcal{O}) = (3, 2, 2, 2, 2, 2, 1, 1)$ which is the 'closest' that the RIV with $\lambda_8 = 1$ of a $(16, 6)$-Oval can be to $(2, 2, 2, 2, 2, 2, 2, 1)$, *i.e.*, Oval $\mathcal{O}$ is the 'closest' that a $(16, 6)$-Oval with one square rhomb can be to a magic $(16, 6, 2)$-Oval. Oval $\mathcal{O}$ has $\lambda_1 = 3$ (instead of $\lambda_1 = 2$ for a magic $(16, 6, 2)$-Oval), and $\lambda_7 = 1$ (instead of $\lambda_7 = 2$). Alternatively, $S = \beta([1\ 1\ 2\ 1\ 5\ 6]) = \{0, 1, 2, 4, 5, 10\}$ is the 'closest' that a 6-subset S' of $\mathbb{Z}_{16}$ with the frequency in $\Delta(S')$ of 8 equal to 2 can be to a $(16, 6, 2)$-CDS. In $\Delta(S)$ the frequencies of 1 and 15 are 3 (instead of 2), and the frequencies of 7 and 9 are 1 (instead of 2).

(n, k, λ)	D	TAIS
(7, 3, 1)	$\{0, 1, 3\}$	[1 2 4]
(11, 5, 2)	$\{0, 1, 2, 6, 9\}$	[1 1 4 3 2]
(13, 4, 1)	$\{0, 1, 3, 9\}$	[1 2 6 4] [1 3 2 7]
(15, 7, 3)	$\{0, 1, 2, 4, 5, 8, 10\}$	[1 1 2 1 3 2 5]
(19, 9, 4)	$\{0, 1, 2, 3, 5, 7, 12, 13, 16\}$	[1 1 1 2 2 5 1 3 3]
(21, 5, 1)	$\{0, 1, 6, 8, 18\}$	[1 5 2 10 3]
(23, 11, 5)	$\{0, 1, 2, 3, 5, 7, 8, 11, 12, 15, 17\}$	[1 1 1 2 2 1 3 1 3 2 6]
(31, 6, 1)	$\{0, 1, 3, 8, 12, 18\}$	[1 2 5 4 6 13] [1 3 6 2 5 14] [1 5 12 4 7 2] [1 7 3 2 4 14] [1 10 8 7 2 3]
(31, 15, 7)–A (31, 15, 7)–B	$\{0, 1, 2, 3, 5, 7, 11, 14, 15, 16, 22, 23, 26, 28, 29\}$ $\{0, 1, 2, 3, 7, 9, 11, 12, 13, 18, 21, 25, 26, 28, 29\}$	[1 1 1 2 2 4 3 1 1 6 1 3 2 1 2] [1 1 1 3 1 2 1 6 4 1 1 2 2 3 2] [1 1 1 4 1 3 6 2 1 1 2 1 2 2 3] [1 1 1 4 2 2 1 1 5 3 4 1 2 1 2]
(35, 17, 8)	$\{0, 1, 2, 3, 5, 6, 10, 16, 17, 18, 22, 24, 25, 27, 28, 31, 33\}$	[1 1 1 2 1 4 6 1 1 4 2 1 2 1 3 2 2]
(37, 9, 2)	$\{0, 1, 3, 7, 17, 24, 25, 29, 35\}$	[1 2 4 10 7 1 4 6 2] [1 3 2 4 5 2 1 7 12]
(40, 13, 4)	$\{0, 1, 2, 4, 5, 8, 13, 14, 17, 19, 24, 26, 34\}$	[1 1 2 1 3 5 1 3 2 5 2 8 6] [1 1 7 1 3 2 1 2 2 4 6 7 3]

Table 4: All non-trivial (n, k, λ)-CDS's (up to (u, z)-equivalence) and the corresponding TAIS's of all non-trivial magic (n, k, λ)-Ovals (up to congruency) for $n \leq 40$ and $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

6 Oval-partitions of $\{2n\}^p$, cyclic difference families, triangle-partitions of $\binom{n}{2}$

See Section 3.9 of Schoen [8] for a preliminary version of some of the research in this Section; see also Schoen and McK Shorb [9].

Let $\mathcal{O}^p$ denote p copies of Oval $\mathcal{O}$, in particular $\{2n\}^p$ denotes p copies of the regular $2n$-gon $\{2n\}$.

Definition 6.1 An *Oval-partition* of $\{2n\}^p$ is a partition of the rhombs from $\{2n\}^p$ into q (n, k_i)-Ovals, $\mathcal{O}_i$, for various $q \geq 1$ and various $k_i \geq 2$:

$$\{2n\}^p \rightarrow \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_q. \tag{4}$$

Clearly (4) is equivalent to

$$p \times \text{RIV}(\{2n\}) = \sum_{i=1}^{q} \text{RIV}(\mathcal{O}_i). \tag{5}$$

We focus on $p = 1$ and sometimes shorten $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_q$ to $\mathcal{O}_1\mathcal{O}_2 \cdots \mathcal{O}_q$.

Remark 6.2 Because the regular $2n$-gon $\{2n\}$ is a magic (n, n, n)-Oval then, along the lines of Theorem 5.11, we can prove that in Oval-partition (4) with $p = 1$ the total number of 1's in the TAIS's of the Ovals in $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_q$ equals n.

Definitions 6.3 distinct Oval-partition, $\mathcal{OP}(n)$, $\mathcal{DOP}(n)$

(1) An Oval-partition is *distinct* if it contains distinct Ovals.

(2) $\mathcal{OP}(n)$ is the total number of Oval-partitions of $\{2n\}$, for $n \geq 2$; we define $\mathcal{OP}(1) = 1$.

(3) $\mathcal{DOP}(n)$ is the total number of distinct Oval-partitions of $\{2n\}$, for $n \geq 2$; we define $\mathcal{DOP}(1) = 1$.

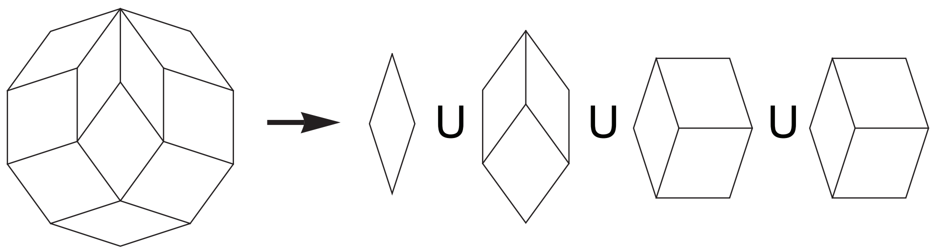
See Table 5 for all Oval-partitions of $\{2n\}$ and the corresponding triangle-partition of $\binom{n}{2}$ (see Section 6.3), for $n = 2, 3, 4,$ and 5.

n	$\binom{n}{2}$	q	O-p of $\{2n\}$	Δ-p of $\binom{n}{2}$	$\mathcal{OP}(n)$	Distinct?	$\mathcal{DOP}(n)$
2	1	1	$\mathcal{O}_1$	1	1	Yes	1
3	3	1	$\mathcal{O}_2$	3	2	Yes	1
3	3	3	$\mathcal{O}_1^3$	1^3		No	
4	6	1	$\mathcal{O}_4$	6	4	Yes	1
4	6	2	$\mathcal{O}_3^2$	3^2		No	
4	6	4	$\mathcal{O}_1^2\mathcal{O}_2\mathcal{O}_3$	$1^3 3$		No	
4	6	6	$\mathcal{O}_1^4\mathcal{O}_2^2$	1^6		No	
5	10	1	$\mathcal{O}_6$	[10]	12	Yes	3
5	10	3	$\mathcal{O}_1\mathcal{O}_4\mathcal{O}_5$	136		Yes	
5	10	3	$\mathcal{O}_2\mathcal{O}_3\mathcal{O}_5$	136		Yes	
5	10	4	$\mathcal{O}_1\mathcal{O}_3\mathcal{O}_4^2$	13^3		No	
5	10	4	$\mathcal{O}_2\mathcal{O}_3^2\mathcal{O}_4$	13^3		No	
5	10	5	$\mathcal{O}_1^2\mathcal{O}_2^2\mathcal{O}_5$	$1^4 6$		No	
5	10	6	$\mathcal{O}_1^3\mathcal{O}_2\mathcal{O}_4^2$	$1^4 3^2$		No	
5	10	6	$\mathcal{O}_1^2\mathcal{O}_2^2\mathcal{O}_3\mathcal{O}_4$	$1^4 3^2$		No	
5	10	6	$\mathcal{O}_1\mathcal{O}_2^3\mathcal{O}_3^2$	$1^4 3^2$		No	
5	10	8	$\mathcal{O}_1^4\mathcal{O}_2^3\mathcal{O}_4$	$1^7 3$		No	
5	10	8	$\mathcal{O}_1^3\mathcal{O}_2^4\mathcal{O}_3$	$1^7 3$		No	
5	10	10	$\mathcal{O}_1^5\mathcal{O}_2^5$	1^{10}		No	

Table 5: All Oval-partitions (O-p) of $\{2n\}$ and the corresponding triangle-partition (Δ-p) of $\binom{n}{2}$ (see Section 6.3); the values of $\mathcal{OP}(n)$ and $\mathcal{DOP}(n)$, for $2 \leq n \leq 5$. The Oval numbering $\mathcal{O}_i$ refers to Table 2.

Example 6.4 $n = 5$. See Fig. 7. As an example with $n = 5$, we check Equation (5) for the Oval-partition $\mathcal{O}_1\mathcal{O}_3\mathcal{O}_4^2$ of $\{10\}$ from Table 5:

$$(5, 5) = (1, 0) + (2, 1) + 2\,(1, 2).$$



$$\{10\} \quad \rightarrow \quad \mathcal{O}([1\ 4]) \cup \mathcal{O}([1\ 1\ 3]) \cup \mathcal{O}([1\ 2\ 2]) \cup \mathcal{O}([1\ 2\ 2])$$

Figure 7: The Oval-partition $\mathcal{O}_1\mathcal{O}_3\mathcal{O}_4^2$ of $\{10\}$.

Observe that the total number of 1's in the TAIS's of the Ovals in the above Oval-partition equals $n = 5$, in agreement with Remark 6.2.

See Table 2, $n = 5$. In total there are 6 $(5, k)$-Ovals: $\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_5, \mathcal{O}_6\}$. Let $\mathcal{RIV}(5) = \{\mathrm{RIV}(\mathcal{O}_1), \mathrm{RIV}(\mathcal{O}_2), \mathrm{RIV}(\mathcal{O}_3), \mathrm{RIV}(\mathcal{O}_4), \mathrm{RIV}(\mathcal{O}_5), \mathrm{RIV}(\mathcal{O}_6)\} = \{(1, 0), (0, 1), (2, 1), (1, 2), (3, 3), (5, 5)\}$. Then to find all Oval-partitions of $\{10\}$ is equivalent to finding all sums of elements of $\mathcal{RIV}(5)$ which are equal to $\mathrm{RIV}(\{10\}) = (5, 5)$, where elements can be used more than once.

Remark 6.5 Similarly, to find all Oval-partitions of $\{2n\}$ is equivalent to finding all sums of elements of the multiset of RIV's of all (n, k)-Ovals which are equal to $\mathrm{RIV}(\{2n\})$, where elements can be used more than once.

The values of $\mathcal{OP}(n)$ and $\mathcal{DOP}(n)$ for $2 \leq n \leq 5$ are given in Table 5, we have also computed $\mathcal{OP}(6) = 58$, $\mathcal{DOP}(6) = 7$, $\mathcal{DOP}(7) = 42$, and $\mathcal{DOP}(8) = 334$. The sequences $\{\mathcal{OP}(n) \,|\, n \geq 1\} = \{1, 1, 2, 4, 12, 58, \ldots\}$ and $\{\mathcal{DOP}(n) \,|\, n \geq 1\} = \{1, 1, 1, 1, 3, 7, 42, 334, \ldots\}$ now appear in [7] as sequences A177921 and A181148 respectively.

We may also think about the Oval-partition $\{2n\} \rightarrow \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_q$ in terms of subsets $S \subseteq \mathbb{Z}_n$. From Example 5.12(a) the regular $2n$-gon $\{2n\}$ is a magic (n, n, n)-Oval with corresponding (n, n, n)-CDS $D = \{0, 1, \ldots, n-1\}$. We modify the proof of Theorem 5.11 to give the following.

Theorem 6.6 *The Oval-partition $\{2n\} \rightarrow \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_q$ exists if and only if there exists q subsets $D_1, D_2, \ldots, D_q \subseteq \mathbb{Z}_n$ with the property that $\Delta(\{0, 1, \ldots, n-1\}) = \Delta(D_1) \cup \Delta(D_2) \cup \cdots \cup \Delta(D_q)$.*

Example 6.7 $n = 5$. See Example 6.4. We have $D = \{0, 1, 2, 3, 4\}$ and $\Delta(D) = \{1^5, 2^5, 3^5, 4^5\}$, and subsets of $\mathbb{Z}_5$: $D_1 = \{0, 1\}$, $D_2 = \{0, 1, 2\}$, and $D_3 = D_4 = \{0, 1, 3\}$.

6.1 Homologous Oval-partitions, isopart triples, cyclic difference families

Here we consider Oval-partitions of $\{2n\}^p$ in which the Ovals $\mathcal{O}_i$ are (n, k)-Ovals, where k is fixed.

Definition 6.8 A *homologous* Oval-partition of $\{2n\}^p$ is a partition of the rhombs from $\{2n\}^p$ into q (n, k)-Ovals, $\mathcal{O}_i$, for a *fixed* $k \geq 2$:

$$\{2n\}^p \to \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_q.$$

Note that the (n, k)-Ovals $\mathcal{O}_i$ need not be congruent.

When $p = 1$ for a homologous Oval-partition of $\{2n\}$ to exist we require $\binom{k}{2} | \binom{n}{2}$. There is a homologous Oval-partition of $\{2n\}$ into $q = 1$ (n, n)-Oval, namely into $\{2n\}$ itself, and another into $q = \binom{n}{2}$ $(n, 2)$-Ovals, namely into the $\binom{n}{2}$ rhombs of $\{2n\}$. We consider these two partitions as trivial, and so in the following restrict ourselves to $2 \leq q \leq \binom{n}{2} - 1$.

Definitions 6.9 $[(n, k), q]$ isopart triple, realizable

(1) The ordered triple $[(n, k), q]$ is an *isopart triple* if

$$\binom{n}{2} = q \binom{k}{2} \quad \text{for some} \quad 2 \leq q \leq \binom{n}{2} - 1,$$

so $k \geq 3$.

(2) The isopart triple $[(n, k), q]$ is *realizable* if there exists a homologous Oval-partition of $\{2n\}$ into q (not necessarily congruent) (n, k)-Ovals.

Example 6.10
(a) $[(n, k), q] = [(4, 3), 2]$. See Table 2. The smallest isopart triple which is realizable is $[(4, 3), 2]$. The relevant homologous Oval-partition is $\{8\} \to \mathcal{O}_3^2 = \mathcal{O}([1\ 1\ 2])^2$.

(b) $[(n,k),q] = [(6,3),5]$. See Table 2. The smallest isopart triple which is not realizable is $[(6,3),5]$.

Suppose there is a homologous Oval-partition

$$\{12\} \to \mathcal{O}_4^{q_1} \cup \mathcal{O}_5^{q_2} \cup \mathcal{O}_6^{q_3}$$

where each $q_i \geq 0$. Then the system of equations containing the equation $q_1 + q_2 + q_3 = 5$ together with the RIV Equations (5):

$$(6,6,3) = q_1(2,1,0) + q_2(1,1,1) + q_3(0,3,0)$$

must have a solution in the non-negative integers. That is, the system

$$q_1 + q_2 + q_3 = 5,\ 2q_1 + q_2 = 6,\ q_1 + q_2 + 3q_3 = 6,\ q_2 = 3,$$

must have a solution in the non-negative integers, a contradiction. Hence the isopart triple $[(6,3),5]$ is not realizable.

See Table 6 for all isopart triples $[(n,k),q]$ for $2 \leq n \leq 16$. All are realizable except $[(\mathbf{6},\mathbf{3}),\mathbf{5}]$ and $[(\mathbf{10},\mathbf{3}),\mathbf{15}]$.

$[(n,k),q]$	Example of a homologous Oval-partition realizing $[(n,k),q]$
$[(4,3),2]$	$\mathcal{O}([1\ 1\ 2])^2$ (magic)
$[(\mathbf{6},\mathbf{3}),\mathbf{5}]$	Not realizable
$[(7,3),7]$	$\mathcal{O}([1\ 2\ 4])^7$ (magic, see Table 4 row $(7,3,1)$, and Example 6.19(b))
$[(9,3),12]$ $[(9,4),6]$	$\mathcal{O}([1\ 1\ 7])^3\,\mathcal{O}([1\ 4\ 4])^3\,\mathcal{O}([2\ 2\ 5])^3\,\mathcal{O}([3\ 3\ 3])^3$ $\mathcal{O}([1\ 1\ 2\ 5])^3\,\mathcal{O}([1\ 3\ 2\ 3])^3$
$[(\mathbf{10},\mathbf{3}),\mathbf{15}]$ $[(10,6),3]$	Not realizable $\mathcal{O}([1\ 1\ 1\ 1\ 3\ 3])\,\mathcal{O}([1\ 1\ 2\ 1\ 1\ 4])\,\mathcal{O}([1\ 2\ 1\ 2\ 2\ 2])$ (see §3.9 p.22 of [8] and Fig. 8)
$[(12,3),22]$ $[(12,4),11]$	$\mathcal{O}([1\ 2\ 9])^4\,\mathcal{O}([1\ 3\ 8])^4\,\mathcal{O}([1\ 4\ 7])^4\,\mathcal{O}([2\ 4\ 6])^4\,\mathcal{O}([2\ 5\ 5])^4\,\mathcal{O}([3\ 3\ 6])^2$ $\mathcal{O}([1\ 1\ 3\ 7])\,\mathcal{O}([1\ 2\ 1\ 8])\,\mathcal{O}([1\ 2\ 4\ 5])\,\mathcal{O}([1\ 2\ 5\ 4])\,\mathcal{O}([1\ 2\ 2\ 7])\,\mathcal{O}([1\ 3\ 1\ 7])$ $\mathcal{O}([1\ 4\ 1\ 6])\,\mathcal{O}([1\ 4\ 2\ 5])\,\mathcal{O}([2\ 2\ 2\ 6])\,\mathcal{O}([2\ 2\ 3\ 5])\,\mathcal{O}([3\ 3\ 3\ 3])$
$[(13,3),26]$ $[(13,4),13]$	$\mathcal{O}([1\ 3\ 9])^{13}\,\mathcal{O}([2\ 5\ 6])^{13}$ $\mathcal{O}([1\ 2\ 6\ 4])^{13}$ (magic, see Table 4 row $(13,4,1)$)
$[(15,3),35]$ $[(15,6),7]$ $[(15,7),5]$	$\mathcal{O}([1\ 1\ 13])^5\,\mathcal{O}([1\ 7\ 7])^5\,\mathcal{O}([2\ 2\ 11])^5\,\mathcal{O}([3\ 3\ 9])^5\,\mathcal{O}([3\ 6\ 6])^5\,\mathcal{O}([4\ 4\ 7])^5\,\mathcal{O}([5\ 5\ 5])^5$ $\mathcal{O}([1\ 1\ 2\ 1\ 6\ 4])\,\mathcal{O}([1\ 1\ 2\ 3\ 2\ 6])\,\mathcal{O}([1\ 1\ 2\ 3\ 6\ 2])\,\mathcal{O}([1\ 2\ 2\ 7\ 1\ 2])$ $\mathcal{O}([1\ 2\ 4\ 1\ 2\ 5])\,\mathcal{O}([1\ 2\ 4\ 1\ 4\ 3])\,\mathcal{O}([1\ 3\ 2\ 4\ 1\ 4])$ $\mathcal{O}([1\ 1\ 2\ 1\ 3\ 2\ 5])^5$ (magic, see Table 4 row $(15,7,3)$, and Example 6.19(c))
$[(16,3),40]$ $[(16,4),20]$ $[(16,5),12]$ $[(16,6),8]$	$\mathcal{O}([1\ 2\ 13])^8\,\mathcal{O}([1\ 7\ 8])^8\,\mathcal{O}([2\ 4\ 10])^8\,\mathcal{O}([3\ 4\ 9])^8\,\mathcal{O}([5\ 5\ 6])^8$ See §3.9 p.23 of [8] See Example 6.11 $\mathcal{O}([1\ 1\ 2\ 1\ 5\ 6])^4\,\mathcal{O}([1\ 5\ 2\ 2\ 3\ 3])^4$ (see Example 6.20)

Table 6: All isopart triples $[(n,k),q]$ for $2 \le n \le 16$, and an example of a homologous Oval-partition realizing the triple. Triples $[(\mathbf{6},\mathbf{3}),\mathbf{5}]$ and $[(\mathbf{10},\mathbf{3}),\mathbf{15}]$ are not realizable.

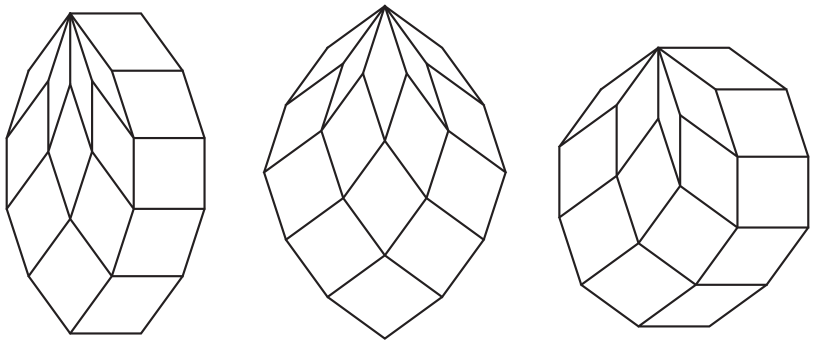


Figure 8: The homologous Oval-partition of $\{20\}$ for isopart triple $[(10, 6), 3]$ from Table 6.

Example 6.11 $(n, k) = (16, 5)$. Isopart triple $[(16, 5), 12]$. See §3.9 p.24 of [8]. Here each of the 12 $(16, 5)$-Ovals are distinct, *i.e.*, incongruent. The Table below gives the TAIS's and RIV's of these 12 Ovals.

TAIS	RIV
[1 1 1 3 10]	(3, 2, 2, 1, 1, 1, 0, 0)
[1 2 9 1 3]	(2, 1, 2, 2, 1, 1, 1, 0)
[1 5 2 3 5]	(1, 1, 1, 0, 3, 2, 1, 1)
[1 4 3 2 6]	(1, 1, 1, 1, 2, 1, 2, 1)
[1 2 5 1 7]	(2, 1, 1, 0, 1, 1, 2, 2)
[2 2 2 3 7]	(0, 3, 1, 2, 1, 1, 2, 0)
[2 2 3 2 7]	(0, 3, 1, 1, 2, 0, 3, 0)
[1 2 3 6 4]	(1, 1, 2, 1, 2, 2, 1, 0)
[1 3 1 3 8]	(2, 0, 2, 3, 1, 0, 1, 1)
[1 1 3 3 8]	(2, 1, 2, 1, 1, 1, 1, 1)
[2 4 2 4 4]	(0, 2, 0, 3, 0, 4, 0, 1)
[1 3 5 1 6]	(2, 0, 1, 1, 1, 2, 2, 1)
	(16,16,16,16,16,16,16, 8)

Homologous Oval-partitions are closely related to another class of combinatorial objects, (*cf.*, Theorem 6.6):

Definition 6.12 A (n, k, λ)-*cyclic difference family* – (n, k, λ)-CDF – is a collection of q k-subsets $D_1, D_2, \ldots, D_q \subseteq \mathbb{Z}_n$ with the property that

$\Delta(D_1) \cup \Delta(D_2) \cup \cdots \cup \Delta(D_q)$ contains every non-zero element of $\mathbb{Z}_n$ exactly λ times.

Remark 6.13 See Equation (3). In a (n, k, λ)-CDF we have

$$\lambda(n-1) = q\,k(k-1),$$

hence $q = \frac{\lambda(n-1)}{k(k-1)}$ is an integer.

From Definition 6.8 of a homologous Oval-partition of $\{2n\}$ and Definition 6.12 of a (n, k, λ)-CDF and Theorem 6.6 we have the following result.

Corollary 6.14 *There exists a homologous Oval-partition of $\{2n\}$ into q (n, k)-Ovals if and only if there exists a (n, k, n)-CDF.*

Clearly, by taking unions of CDF's, there exists a (n, k, n)-CDF if and only if there exists a collection of (n, k, λ_i)-CDF's with $\sum_i \lambda_i = n$. Hence, another main result follows.

Theorem 6.15 *There exists a homologous Oval-partition of $\{2n\}$ into q (n, k)-Ovals (i.e., isopart triple $[(n, k), q]$ is realizable) if and only if there exists a collection of (n, k, λ_i)-CDF's with $\sum_i \lambda_i = n$.*

Example 6.16
(a) $(n, k) = (9, 4)$. See Example 1.6(a) p.470 of [3] for the $(9, 4, 3)$-CDF with $D_1 = \{0, 1, 2, 4\}$ and $D_2 = \{0, 3, 4, 7\}$. Using 3 copies of this CDF we produce the following homologous Oval-partition of $\{18\}$ into 6 $(9, 4)$-Ovals: $\mathcal{O}(\alpha(D_1))^3\,\mathcal{O}(\alpha(D_2))^3 = \mathcal{O}([1\ 1\ 2\ 5])^3\,\mathcal{O}([1\ 3\ 2\ 3])^3$. This realizes isopart triple $[(9, 4), 6]$ with the same partition as given in Table 6.

(b) $(n, k) = (16, 3)$. Conversely, we may take a partition which realizes an isopart triple from Table 6 and produce a CDF. For example, the 5 $(16, 3)$-Ovals from row $[(16, 3), 40]$: $\mathcal{O}([1\,2\,13])\,\mathcal{O}([1\,7\,8])\,\mathcal{O}([2\,4\,10])\,\mathcal{O}([3\,4\,9])\,\mathcal{O}([5\,5\,6])$ produce a $(16, 3, 2)$-CDF with $D_1 = \{0, 1, 3\}, D_2 = \{0, 1, 8\}, D_3 = \{0, 2, 6\}$, $D_4 = \{0, 3, 7\}$, and $D_5 = \{0, 5, 10\}$ which is not (u, z)-equivalent to the $(16, 3, 2)$-CDF in Examples 16.13, p.394 of Colbourn and Dinitz [4].

(c) $(n, k) = (6, 3)$. From Table 6 we see that isopart triple $[(\mathbf{6}, \mathbf{3}), \mathbf{5}]$ is not realizable, so, from Theorem 6.15, there does not exist a $(6, 3, 6)$-CDF nor a $(6, 3, 2)$-CDF; see Table II.2.29, p.61 of [4].

(d) $(n, k) = (10, 3)$. Similarly, isopart triple $[(\mathbf{10}, \mathbf{3}), \mathbf{15}]$ is not realizable, so there does not exist a $(10, 3, 10)$-CDF nor a $(10, 3, 2)$-CDF; see Table II.2.29, p.61 of [4] again.

6.2 Magic Oval-partitions

Recall that in a (n, k, λ)-CDS we have $\lambda(n-1) = k(k-1)$.

As mentioned in Section 1 this research was partially motivated by Question (iii) on p. 10 of Schoen [8].

Fix $n \geq 2$, for which integers p and q can the rhombs contained in p copies of $\{2n\}$ be partitioned to tile q congruent *Ovals?*

Definition 6.17 A *magic* Oval-partition of $\{2n\}^p$ is a partition of the rhombs contained in $\{2n\}^p$ into q *congruent* (n, k)-Ovals, $\mathcal{O}$:

$$\{2n\}^p \to \mathcal{O}^q. \tag{6}$$

We now show that if such a magic Oval-partition of $\{2n\}^p$ exists, then $\mathcal{O}$ is magic.

Theorem 6.18 *The partition $\{2n\}^p \to \mathcal{O}^q$ exists if and only if there exists a $(n, k, \frac{pn}{q})$-CDS, ($\mathcal{O}$ will then be a magic $(n, k, \frac{pn}{q})$-Oval).*

Proof. For odd n. Necessity: suppose that such a partition (6) exists. Consider ρ_h, the rhomb of SRI_{2n} with principle index h, for any fixed $h = 1, 2, \ldots, \frac{n-1}{2}$. It appears pn times on the left in partition (6) and $q\lambda_h$ times on the right, *i.e.*, it appears $\lambda_h = \frac{pn}{q}$ times in $\mathcal{O}$. Thus λ_h is independent of h, and so $\mathcal{O}$ is a magic $(n, k, \frac{pn}{q})$-Oval, (for some suitable k satisfying $k(k-1) = \frac{pn}{q}(n-1)$).
Sufficiency: conversely given a magic $(n, k, \frac{pn}{q})$-Oval $\mathcal{O}$ it contains $\frac{pn}{q}$ copies of each rhomb ρ_h. So $\mathcal{O}^q$ contains pn copies of each ρ_h, but this is exactly the number of copies of ρ_h in $\{2n\}^p$.

For even n. The proof is similar to the above, but we consider the non-square rhombs ρ_h for $h = 1, 2, \ldots, \frac{n}{2}-1$, and the square rhomb $\rho_{\frac{n}{2}}$ as separate cases. □

We can find a partition where p and q are the smallest by considering:

$$\frac{p}{q} = \frac{\lambda}{n} = \frac{\lambda^*}{n^*}$$

where $\gcd(\lambda^*, n^*) = 1$. This gives the partition:

$$\{2n\}^{\lambda^*} \to \mathcal{O}^{n^*}.$$

Any other partition with the same $\mathcal{O}$ is a 'multiple' of this one.

Note that if $\lambda^* = 1$ and $2 \le n^* \le \binom{n}{2} - 1$ then $[(n,k), n^*]$ is a realizable isopart triple.

Example 6.19

(a) See Examples 5.12(a) and (b). Oval $\{2n\}'$ is a magic $(n, n-1, n-2)$-Oval obtained from the regular $2n$-gon $\{2n\}$ by removing its right-hand strip of rhombs. For odd n we have $\frac{\lambda}{n} = \frac{n-2}{n} = \frac{\lambda^*}{n^*}$, so the smallest magic Oval-partition is

$$\{2n\}^{n-2} \to \{2n\}'^{\,n}.$$

For even $n = 2m$ the smallest magic Oval-partition is

$$\{2n\}^{m-1} \to \{2n\}'^{\,m}.$$

(b) See Example 5.12(c). Oval $\mathcal{O}([1\ 2\ 4])$ is a magic $(7,3,1)$-Oval with RIV $(1,1,1)$. Now $\frac{\lambda}{n} = \frac{1}{7} = \frac{\lambda^*}{n^*}$, so we have the following magic Oval-partition

$$\{14\}^1 \to \mathcal{O}([1\ 2\ 4])^7.$$

The decomposition of $1 \times \mathrm{RIV}(\{14\})$ is $1 \times (7,7,7) \to 7 \times (1,1,1)$, and the relevant realizable isopart triple is $[(7,3),7]$; see Table 6.

(c) $(n,k) = (15,7)$. See Example 5.12(d). Oval $\mathcal{O}([1\ 1\ 2\ 1\ 3\ 2\ 5])$ is a magic $(15,7,3)$-Oval. Here $\frac{\lambda}{n} = \frac{3}{15} = \frac{1}{5}$ so $\lambda^* = 1$ and $n^* = 5$, this gives

$$\{30\}^1 \to \mathcal{O}([1\ 1\ 2\ 1\ 3\ 2\ 5])^5.$$

The RIV decomposition is $1 \times (15,15,15,15,15,15,15) \to 5 \times (3,3,3,3,3,3,3)$ and $[(15,7),5]$ is the corresponding realizable isopart triple.

(d) $(n,k) = (11,5)$. The $(11,5)$-Oval $\mathcal{O}([1\ 1\ 4\ 3\ 2])$ is a magic $(11,5,2)$-Oval. Here $\frac{\lambda}{n} = \frac{2}{11}$ so $\lambda^* = 2$ and $n^* = 11$. This gives us the following magic Oval-partition where $p \ne 1$:

$$\{22\}^2 \to \mathcal{O}([1\ 1\ 4\ 3\ 2])^{11}.$$

The RIV decomposition is $2 \times (11,11,11,11,11) \to 11 \times (2,2,2,2,2)$.

Example 6.20 $(n,k) = (16,6)$. From Example 5.21 there does not exist a magic $(16,6,2)$-Oval, *i.e.*, there does not exist a $(16,6)$-Oval with RIV $(2,2,2,2,2,2,2,1)$. Now $\mathrm{RIV}(\{16\}) = (16,16,16,16,16,16,16,8)$, so $\{16\} \not\to$

$\mathcal{O}^8$ where $\mathcal{O}$ is a fixed $(16,6)$-Oval. In row $[(16,6),8]$ of Table 6 we gave the homologous Oval-partition

$$\{16\} \to \mathcal{O}([1\ 1\ 2\ 1\ 5\ 6])^4\, \mathcal{O}([1\ 5\ 2\ 2\ 3\ 3])^4,$$

with RIV decomposition

$$(16,16,16,16,16,16,16,8) = 4\,(3,2,2,2,2,2,1,1) + 4\,(1,2,2,2,2,2,3,1).$$

We now show that for *every* homologous Oval-partition $\{16\} \to \mathcal{O}_1^{q_1}\, \mathcal{O}_2^{q_2}$ into exactly 2 incongruent $(16,6)$-Ovals $\mathcal{O}_1$ and $\mathcal{O}_2$, we have $q_1 = q_2 = 4$.

Suppose $q_1 = 1$ and $q_2 = 7$. Let $\mathrm{RIV}(\mathcal{O}_1) = (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8)$ and $\mathrm{RIV}(\mathcal{O}_2) = (\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7,\mu_8)$. Then

$$\begin{aligned}&(16,16,16,16,16,16,16,8)\\ &\quad = (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8) + 7\,(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7,\mu_8),\end{aligned}$$

and $\lambda_h + 7\mu_h = 16$ for $h = 1,2,\ldots,7$. Hence for a fixed $h = 1,2,\ldots,7$ we have either $\lambda_h = \mu_h = 2$, or $\lambda_h = 9$ and $\mu_h = 1$, or $\lambda_h = 16$ and $\mu_h = 0$. In particular $\lambda_h \geq 2$ for every $h = 1,2,\ldots,7$. Now $\mathcal{O}_1$ is a $(16,6)$-Oval so $\sum_{h=1}^{8} \lambda_h = \binom{6}{2} = 15$. Thus if $\lambda_h = 2$ for every $h = 1,2,\ldots,7$ then $\lambda_8 = 1$ and $\mathcal{O}_1$ is a magic $(16,6,2)$-Oval, a contradiction. Hence for some h with $h = 1,2,\ldots,7$ we must have $\lambda_h = 9$ or $\lambda_h = 16$, so $\sum_{h=1}^{7} \lambda_h \geq 6\times 2+9 = 21$. But $\sum_{h=1}^{7} \lambda_h \leq 15$, a contradiction. Hence there is no homologous Oval-partition $\{16\} \to \mathcal{O}_1^1\, \mathcal{O}_2^7$. Similarly, the other possible homologous Oval-partitions $\{16\} \to \mathcal{O}_1^2\, \mathcal{O}_2^6$ or $\{16\} \to \mathcal{O}_1^3\, \mathcal{O}_2^5$ do not exist. Hence the only homologous Oval-partition $\{16\} \to \mathcal{O}_1^{q_1}\, \mathcal{O}_2^{q_2}$ has $q_1 = q_2 = 4$; an explicit example is given above.

6.3 Triangular-partitions of $\binom{n}{2}$

Recall the *triangular numbers*: $\{\binom{n}{2}, n \geq 2\} = \{1,3,6,10,15,21,28,\ldots\}$.

Definitions 6.21 Triangular-partition (Δ-partition) of $\binom{n}{2}$, realizable

(1) A *triangular-partition* (Δ*-partition*) of $\binom{n}{2}$ is an integer partition of $\binom{n}{2}$ with each part a triangular number.

(2) A Δ-partition of $\binom{n}{2}$ with q parts in which the i-th part is $\binom{k_i}{2}$ is *realizable* if there exists an Oval-partition of $\{2n\}$ into q Ovals $\mathcal{O}_i$ in which $\mathcal{O}_i$ is a (n,k_i)-Oval, for each $i = 1,2,\ldots,q$.

Remark 6.22 The Δ-partition of $\binom{n}{2}$ corresponding to isopart triple $[(n,k),q]$ is $\binom{k}{2}^q$.

Table 7 lists all Δ-partitions of $\binom{n}{2}$ for $n = 2, 3, \ldots, 8$. For a fixed n the Δ-partitions are given with increasing q, and then in lexicographic order for constant q. The Δ-partition $\mathbf{3^5}$ of $\binom{6}{2} = 15$ is the only Δ-partition in Table 7 which is not realizable; see Example 6.10(b), and row $[(\mathbf{6}, \mathbf{3}), \mathbf{5}]$ of Table 6.

n	$\binom{n}{2}$	Δ-partitions of $\binom{n}{2}$
2	1	1
3	3	$3,\ 1^3$
4	6	$6,\ 3^2,\ 1^3 3,\ 1^6$
5	10	$[10],\ 136,\ 13^3,\ 1^4 6,\ 1^4 3^2,\ 1^7 3,\ 1^{10}$
6	15	$[15],\ 36^2,\ 1^2 3[10],\ 3^3 6,\ 1^3 6^2,\ \mathbf{3^5},\ 1^5[10],\ 1^3 3^2 6,\ 1^3 3^4,\ 1^6 36,\ 1^6 3^3,\ 1^9 6,\ 1^9 3^2,\ 1^{12} 3,\ 1^{15}$
7	21	$[21], 6[15],\ 1[10]^2,\ 3^2[15],\ 36^3,\ 1^3 3[15],\ 1^2 3\,6[10],\ 3^3 6^2,\ 1^3 6^3,\ 1^2 3^3[10],\ 3^5 6,\ 1^6[15],\ 1^5 6[10],\ 1^3 3^2 6^2,\ 3^7,\ 1^5 3^2[10],\ 1^3 3^4 6,\ 1^6 36^2,\ 1^3 3^6,\ 1^8 3[10],\ 1^6 3^3 6,\ 1^9 6^2,\ 1^6 3^5,\ 1^{11}[10],\ 1^9 3^2 6,\ 1^9 3^4,\ 1^{12} 36,\ 1^{12} 3^3,\ 1^{15} 6,\ 1^{15} 3^2,\ 1^{18} 3,\ 1^{21}$
8	28	$[28], 16[21],\ 3[10][15],\ 13^2[21],\ 16^2[15],\ 6^3[10],\ 1^3[10][15],\ 1^2 6[10]^2,\ 13^2 6[15],\ 3^2 6^2[10],\ 1^4 3[21],\ 1^2 3^2[10]^2,\ 13^4[15],\ 136^4,\ 3^4 6[10],\ 1^4 36[15],\ 1^3 36^2[10],\ 13^3 6^3,\ 3^6[10],\ 1^7[21],\ 1^5 3[10]^2,\ 1^4 3^3[15],\ 1^4 6^4,\ 1^3 3^3 6[10],\ 13^5 6^2,\ 1^7 6[15],\ 1^6 6^2[10],\ 1^4 3^2 6^3,\ 1^3 3^5[10],\ 13^7 6,\ 1^8[10]^2,\ 1^7 3^2[15],\ 1^6 3^2 6[10],\ 1^4 3^4 6^2,\ 13^9,\ 1^7 36^3,\ 1^6 3^4[10],\ 1^4 3^6 6,\ 1^{10} 3[15],\ 1^9 36[10],\ 1^7 3^3 6^2,\ 1^4 3^8,\ 1^{10} 6^3,\ 1^9 3^3[10],\ 1^7 3^5 6,\ 1^{13}[15],\ 1^{12} 6[10],\ 1^{10} 3^2 6^2,\ 1^7 3^7,\ 1^{12} 3^2[10],\ 1^{10} 3^4 6,\ 1^{13} 36^2,\ 1^{10} 3^6,\ 1^{15} 3[10],\ 1^{13} 3^3 6,\ 1^{16} 6^2,\ 1^{13} 3^5,\ 1^{18}[10],\ 1^{16} 3^2 6,\ 1^{16} 3^4,\ 1^{19} 36,\ 1^{19} 3^3,\ 1^{22} 6,\ 1^{22} 3^2,\ 1^{25} 3,\ 1^{28}$

Table 7: All Δ-partitions of $\binom{n}{2}$ for $2 \leq n \leq 8$. All are realizable except $\mathbf{3^5}$, for $n = 6$.

Example 6.23 $2 \le n \le 6$. See Table 5 for realizations of all Δ-partitions of $\binom{n}{2}$ for $2 \le n \le 5$. See Table 8 for all Δ-partitions of $\binom{6}{2} = 15$ and, except for $\mathbf{3^5}$, an Oval-partition of $\{12\}$ which realizes it. The Δ-partition $\mathbf{3^5}$ is not realizable. The Oval numbering $\mathcal{O}_i$ refers to Table 2.

Δ-p of $\binom{6}{2}$	O-p of $\{12\}$	Δ-p of $\binom{6}{2}$	O-p of $\{12\}$	Δ-p of $\binom{6}{2}$	O-p of $\{12\}$
$[15]$	$\mathcal{O}_{11}$	$\mathbf{3^5}$	Not realizable	$1^6 3^3$	$\mathcal{O}_2^3\mathcal{O}_3^3\mathcal{O}_4^3$
$3\,6^2$	$\mathcal{O}_4\mathcal{O}_8\mathcal{O}_9$	$1^5[10]$	$\mathcal{O}_1^2\mathcal{O}_2^2\mathcal{O}_3\mathcal{O}_{10}$	$1^9 6$	$\mathcal{O}_1^3\mathcal{O}_2^4\mathcal{O}_3^2\mathcal{O}_7$
$1^2 3[10]$	$\mathcal{O}_1\mathcal{O}_2\mathcal{O}_5\mathcal{O}_{10}$	$1^3 3^2 6$	$\mathcal{O}_2\mathcal{O}_3^2\mathcal{O}_4^2\mathcal{O}_8$	$1^9 3^2$	$\mathcal{O}_1^2\mathcal{O}_2^4\mathcal{O}_3^3\mathcal{O}_4^2$
$3^3 6$	$\mathcal{O}_4\mathcal{O}_5^2\mathcal{O}_8$	$1^3 3^4$	$\mathcal{O}_3^3\mathcal{O}_4^3\mathcal{O}_6$	$1^{12} 3$	$\mathcal{O}_1^4\mathcal{O}_2^5\mathcal{O}_3^3\mathcal{O}_4$
$1^3 6^2$	$\mathcal{O}_2^2\mathcal{O}_3\mathcal{O}_7^2$	$1^6 3\,6$	$\mathcal{O}_1\mathcal{O}_2^3\mathcal{O}_3^2\mathcal{O}_4\mathcal{O}_7$	1^{15}	$\mathcal{O}_1^6\mathcal{O}_2^6\mathcal{O}_3^3$

Table 8: All Δ-partitions (Δ-p) of $\binom{6}{2} = 15$ and, except for $\mathbf{3^5}$, an Oval-partition (O-p) of $\{12\}$ which realizes it.

We have extended our results on Δ-partitions of $\binom{n}{2}$ up to $n = 10$.

Example 6.24 For $n = 2, 3, \ldots, 10$ all Δ-partitions of $\binom{n}{2}$ are realizable except $\mathbf{3^5}$ for $n = 6$ (see Examples 6.10(b) and 6.16(c)), and $\mathbf{3^{15}}, \mathbf{3^8[21]}, \mathbf{3^5[10]^3}, \mathbf{3^3[36]}$, and $\mathbf{3[21]^2}$ for $n = 10$. The unrealizable Δ-partitions for $n = 10$ were shown to be unrealizable along the lines of Example 6.10(b) using MAPLE; see also Example 6.16(d).

7 u-equivalent Ovals

In this Section we explain why 2 incongruent (n,k)-Ovals can have RIV's that are permutations of each other. For example, see Table 2 $n = 7$, there are 4 incongruent $(7,3)$-Ovals: $\{\mathcal{O}_4, \mathcal{O}_5, \mathcal{O}_6, \mathcal{O}_7\}$, but 3 of them: $\{\mathcal{O}_4, \mathcal{O}_6, \mathcal{O}_7\}$ have RIV's that are permutations of $(2, 1, 0)$.

Recall the operations α and β from Definitions 2.8, and the function r from Equation (2). Recall also that $S = \{s_1, s_2, \ldots, s_k\}$ where $0 \le s_1 < s_2 < \cdots < s_k$ is a k-subset of $\mathbb{Z}_n$ with elements in increasing order. For $u \in U(n)$, when we form $uS = \{us_1, us_2, \ldots, us_k\}$ we will always rearrange the elements of uS in increasing order also, so that we may apply α to uS.

Further, we let $[\lfloor \frac{n}{2} \rfloor] = \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$.

Lemma 7.1 *Let principal index h occur λ_h times in OIT$(\alpha(S)) = r(\delta(S))$. Then for any $u \in U(n)$ principal index uh occurs λ_h times in OIT$(\alpha(uS)) = r(\delta(uS))$.*

Proof. Let principal index uh occur λ_{uh} times in $\mathrm{OIT}(\alpha(uS)) = r(\delta(uS))$. We must show that $\lambda_h = \lambda_{uh}$.

First we show $\lambda_h \leq \lambda_{uh}$: principal index h occurs λ_h times in $\mathrm{OIT}(\alpha(S)) = r(\delta(S))$, so there are λ_h pairs $\{s_j, s_i\}$ where $1 \leq i < j \leq k$ for which $s_j - s_i \in \{h, -h\}$. Consider $uS = \{us_1, us_2, \ldots, us_k\} = \{v_1, v_2, \ldots, v_k\}$ where $0 \leq v_1 < v_2 < \cdots < v_k$. Suppose pair $\{s_j, s_i\}$ satisfies $s_j - s_i \in \{h, -h\}$ with $s_j - s_i = h$. Then $us_j - us_i = uh$, *i.e.*, $v_\ell - v_{\ell'} = uh$ where $v_\ell = us_j$ and $v_{\ell'} = us_i$. If $\ell > \ell'$ then pair $\{v_\ell, v_{\ell'}\}$ satisfies $v_\ell - v_{\ell'} = uh$ and so $v_\ell - v_{\ell'} \in \{uh, -uh\}$ and $1 \leq \ell' < \ell \leq k$, and if $\ell < \ell'$ then pair $\{v_{\ell'}, v_\ell\}$ satisfies $v_{\ell'} - v_\ell = -uh$ and so again $v_{\ell'} - v_\ell \in \{uh, -uh\}$ and $1 \leq \ell < \ell' \leq k$. Thus, in either case, a pair $\{s_j, s_i\}$ for which $s_j - s_i = h$ where $1 \leq i < j \leq k$ gives rise to a pair $\{v_a, v_b\}$ for which $v_a - v_b \in \{uh, -uh\}$ and $1 \leq a < b \leq k$. Similarly if $s_j - s_i = -h$. Thus $\lambda_h \leq \lambda_{uh}$.

To show that $\lambda_h \geq \lambda_{uh}$, *i.e.*, $\lambda_{uh} \leq \lambda_h$ we start with $V = uS = \{us_1, us_2, \ldots, us_k\} = \{v_1, v_2, \ldots, v_k\}$ and argue as above with u replaced by u^{-1}.

The above two paragraphs give $\lambda_h = \lambda_{uh}$ as required. □

Definitions 7.2 $u\mathcal{O}$, permutation P_u

Let $\mathcal{O}$ be an (n,k)-Oval with TAIS T, and let $u \in U(n)$.

(1) $u\mathcal{O}$ is the (n,k)-Oval with TAIS $\alpha(u\beta(T))$.

(2) *Permutation* P_u is the permutation of $[\lfloor \frac{n}{2} \rfloor]$ given by $P_u(h) = r(uh)$, for every $h \in [\lfloor \frac{n}{2} \rfloor]$ and $u \in U(n)$.

Theorem 7.3 *Let $\mathcal{O}$ be an (n,k)-Oval and let $u \in U(n)$. Then* $\mathrm{RIV}(u\mathcal{O}) = P_u(\mathrm{RIV}(\mathcal{O}))$.

Proof. For each $h \in [\lfloor \frac{n}{2} \rfloor]$ let the h-th entry of $\mathrm{RIV}(\mathcal{O})$ be λ_h then, from Lemma 7.1, the uh-th entry of $\mathrm{RIV}(u\mathcal{O})$ is also λ_h. Hence $\mathrm{RIV}(u\mathcal{O})$ is a permutation of $\mathrm{RIV}(\mathcal{O})$ where, for each $h \in [\lfloor \frac{n}{2} \rfloor]$, the h-th entry (of $\mathrm{RIV}(\mathcal{O})$) is moved to the uh-th entry (of $\mathrm{RIV}(u\mathcal{O})$), *i.e.*, is moved by the application of permutation P_u. Thus the result. □

Example 7.4
(a) For every $n \geq 2$ we have $-1 \in U(n)$ and P_{-1} is the identity permutation of $[\lfloor \frac{n}{2} \rfloor]$. Hence $\mathrm{RIV}(-\mathcal{O}) = \mathrm{RIV}(\mathcal{O})$. Confirming this, see Lemma 3.2(i), we have $\mathrm{TAIS}(-\mathcal{O}) \equiv_{\mathrm{cyc}} \overleftarrow{\mathrm{TAIS}}(\mathcal{O})$ and hence $\mathrm{RIV}(-\mathcal{O}) = \mathrm{RIV}(\mathcal{O})$.

(b) $(n,k) = (15,6)$. See Example 2.5. For the $(15,6)$-Oval $\mathcal{X}$ with TAIS $T = [4\ 3\ 2\ 1\ 4\ 1]$ we have $X = \beta(T) = \{0,4,7,9,10,14\}$. Unit $2 \in U(15)$ gives permutation $P_2 = (1\ 2\ 4\ 7)(3\ 6)(5)$ of $[7]$. Now $2X = \{0,3,5,8,13,14\}$, and so $2\mathcal{X} = \mathcal{O}([3\ 2\ 3\ 5\ 1\ 1])$. We check: $\mathrm{RIV}(2\mathcal{X}) = P_2(\mathrm{RIV}(\mathcal{X})) = P_2(2,1,2,2,4,2,2) = (2,2,2,1,4,2,2)$, as required by Theorem 7.3.

(c) $(n,k) = (16,6)$. We show how we used Theorem 7.3 in Example 6.20. In Example 6.20 it was required to find 2 $(16,6)$-Ovals $\mathcal{O}_1$ and $\mathcal{O}_2$ for which $\mathrm{RIV}(\mathcal{O}_1) + \mathrm{RIV}(\mathcal{O}_2) = (4,4,4,4,4,4,4,2)$. From Example 5.21 we had a $(16,6)$-Oval $\mathcal{O} = \mathcal{O}([1\ 1\ 2\ 1\ 5\ 6])$ with $\mathrm{RIV}(\mathcal{O}) = (3,2,2,2,2,2,1,1)$. We observed that $(4,4,4,4,4,4,4,2) - \mathrm{RIV}(\mathcal{O}) = (1,2,2,2,2,2,3,1)$ is a permutation of $\mathrm{RIV}(\mathcal{O})$. Further, unit $7 \in U(16)$ gives permutation $P_7 = (1\ 7)(3\ 5)(2)(4)(6)(8)$ of $[8]$, and $P_7(\mathrm{RIV}(\mathcal{O})) = (1,2,2,2,2,2,3,1)$. Then letting $\mathcal{O}_1 = \mathcal{O}$ and $\mathcal{O}_2 = 7\mathcal{O} = \mathcal{O}([1\ 5\ 2\ 2\ 3\ 3])$ gave the required Ovals.

Definition 7.5 Two (n,k)-Ovals $\mathcal{O}_1$ and $\mathcal{O}_2$ are *u-equivalent*, $\mathcal{O}_1 \equiv_u \mathcal{O}_2$, if there is a $u \in U(n)$ such that $\mathcal{O}_1 = u\mathcal{O}_2$.

It is clear that u-equivalence is an equivalence relation on $\mathcal{O}^*_{\mathrm{c}}(n,k)$, the set of (n,k)-Ovals up to congruency.

Definitions 7.6 $\mathcal{O}^*_{\mathrm{c},\equiv_u}(n,k)$, $\mathcal{O}_{\mathrm{c},\equiv_u}(n,k)$

(1) $\mathcal{O}^*_{\mathrm{c},\equiv_u}(n,k)$ is the set of equivalence classes of $\equiv_u$ in $\mathcal{O}^*_{\mathrm{c}}(n,k)$.

(2) $\mathcal{O}_{\mathrm{c},\equiv_u}(n,k) = |\mathcal{O}^*_{\mathrm{c},\equiv_u}(n,k)|$ is the number of equivalence classes of $\equiv_u$ in $\mathcal{O}^*_{\mathrm{c}}(n,k)$.

Example 7.7 $(n,k) = (7,3)$. See Table 2, $n = 7$. Here $\mathcal{O}_4 = 2\mathcal{O}_6 = 4\mathcal{O}_7$, and $\mathcal{O}_5 = u\mathcal{O}_5$ for every $u \in U(7)$. Hence there are $\mathcal{O}_{\mathrm{c},\equiv_u}(7,3) = 2$ $\equiv_u$-equivalence classes in $\mathcal{O}^*_{\mathrm{c}}(7,3)$, namely $[\mathcal{O}_4] = \{\mathcal{O}_4, \mathcal{O}_6, \mathcal{O}_7\}$ and $[\mathcal{O}_5] = \{\mathcal{O}_5\}$. We have $\mathcal{O}^*_{\mathrm{c},\equiv_u}(7,3) = \{[\mathcal{O}_4], [\mathcal{O}_5]\}$. We say that there are 2 $(7,3)$-Ovals up to u-equivalence, namely Ovals $\mathcal{O}_4$ and $\mathcal{O}_5$; see Table 9.

n	k	$\mathcal{O}_{c,\equiv_u}(n,k)$	$\mathcal{O}^*_{c,\equiv_u}(n,k)$
2	2	1	$\mathcal{O}_1$
3	2	1	$\mathcal{O}_1$
3	3	1	$\mathcal{O}_2$
4	2	2	$\mathcal{O}_1, \mathcal{O}_2$
4	3	1	$\mathcal{O}_3$
4	4	1	$\mathcal{O}_4$
5	2	1	$\mathcal{O}_1$
5	3	1	$\mathcal{O}_3$
5	4	1	$\mathcal{O}_5$
5	5	1	$\mathcal{O}_6$
6	2	3	$\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$
6	3	3	$\mathcal{O}_4, \mathcal{O}_5, \mathcal{O}_6$
6	4	3	$\mathcal{O}_7, \mathcal{O}_8, \mathcal{O}_9$
6	5	1	$\mathcal{O}_{10}$
6	6	1	$\mathcal{O}_{11}$
7	2	1	$\mathcal{O}_1$
7	3	2	$\mathcal{O}_4, \mathcal{O}_5$
7	4	2	$\mathcal{O}_8, \mathcal{O}_9$
7	5	1	$\mathcal{O}_{12}$
7	6	1	$\mathcal{O}_{15}$
7	7	1	$\mathcal{O}_{16}$

n	k	$\mathcal{O}_{c,\equiv_u}(n,k)$	$\mathcal{O}^*_{c,\equiv_u}(n,k)$
8	2	3	$\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_4$
8	3	4	$\mathcal{O}_5, \mathcal{O}_6, \mathcal{O}_7, \mathcal{O}_8$
8	4	6	$\mathcal{O}_{10}, \mathcal{O}_{11}, \mathcal{O}_{12}, \mathcal{O}_{13}, \mathcal{O}_{16}, \mathcal{O}_{17}$
8	5	4	$\mathcal{O}_{18}, \mathcal{O}_{19}, \mathcal{O}_{20}, \mathcal{O}_{21}$
8	6	3	$\mathcal{O}_{23}, \mathcal{O}_{24}, \mathcal{O}_{26}$
8	7	1	$\mathcal{O}_{27}$
8	8	1	$\mathcal{O}_{28}$
9	2	2	$\mathcal{O}_1, \mathcal{O}_3$
9	3	3	$\mathcal{O}_5, \mathcal{O}_6, \mathcal{O}_{11}$
9	4	4	$\mathcal{O}_{12}, \mathcal{O}_{13}, \mathcal{O}_{15}, \mathcal{O}_{17}$
9	5	4	$\mathcal{O}_{22}, \mathcal{O}_{23}, \mathcal{O}_{24}, \mathcal{O}_{29}$
9	6	3	$\mathcal{O}_{32}, \mathcal{O}_{33}, \mathcal{O}_{38}$
9	7	2	$\mathcal{O}_{39}, \mathcal{O}_{41}$
9	8	1	$\mathcal{O}_{43}$
9	9	1	$\mathcal{O}_{44}$
10	2	3	$\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_5$
10	3	4	$\mathcal{O}_6, \mathcal{O}_7, \mathcal{O}_9, \mathcal{O}_{10}$
10	4	9	$\mathcal{O}_{14}, \mathcal{O}_{15}, \mathcal{O}_{16}, \mathcal{O}_{17}, \mathcal{O}_{18}, \mathcal{O}_{19}, \mathcal{O}_{22}, \mathcal{O}_{26}, \mathcal{O}_{27}$
10	5	9	$\mathcal{O}_{30}, \mathcal{O}_{31}, \mathcal{O}_{32}, \mathcal{O}_{33}, \mathcal{O}_{34}, \mathcal{O}_{36}, \mathcal{O}_{37}, \mathcal{O}_{38}, \mathcal{O}_{45}$
10	6	9	$\mathcal{O}_{46}, \mathcal{O}_{47}, \mathcal{O}_{48}, \mathcal{O}_{49}, \mathcal{O}_{50}, \mathcal{O}_{51}, \mathcal{O}_{53}, \mathcal{O}_{57}, \mathcal{O}_{58}$
10	7	4	$\mathcal{O}_{62}, \mathcal{O}_{63}, \mathcal{O}_{65}, \mathcal{O}_{66}$
10	8	3	$\mathcal{O}_{70}, \mathcal{O}_{71}, \mathcal{O}_{74}$
10	9	1	$\mathcal{O}_{75}$
10	10	1	$\mathcal{O}_{76}$

Table 9: All (n,k)-Ovals up to u-equivalence for $2 \le n \le 10$. The equivalence class $[\mathcal{O}_i]$ is denoted by $\mathcal{O}_i$; see Example 7.7.

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