Double Arrays, Triple Arrays, and Balanced Grids with $v = r + c - 1$

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Abstract

In Theorem 6.1 of [3] it was shown that, when $v = r + c - 1$, every triple array $TA(v, k, \lambda_{rr}, \lambda_{cc}, k : r \times c)$ is a balanced grid $BG(v, k, k : r \times c)$. Here we prove the converse of this Theorem. Our final result is: Let $v = r + c - 1$. Then every triple array is a $TA(v, k, c - k, r - k, k : r \times c)$ and every balanced grid is a $BG(v, k, k : r \times c)$, and they are equivalent.

Keywords: arrays, double arrays, triple arrays, balanced grids, designs

1 Introduction, Main Result

We briefly introduce the main players: arrays, double arrays, triple arrays, and balanced grids. See [3] for more details.

Consider a rectangle with r rows and c columns, in which each cell contains exactly one element from the set $V = \{1, 2, \ldots, v\}$. Suppose that the rectangle is *binary*, *i.e.*, every row contains distinct elements and every column contains distinct elements. Further, suppose that the rectangle is *equireplicate*, *i.e.*, every element of V occurs exactly k times in the rectangle for some $k \geq 1$. Call such a rectangle a $r \times c$ *array* based on the set V, and denote it by $\mathcal{A} = A(v, k : r \times c)$.

An array $\mathcal{A}$ is a *double array* if it satisfies the following two properties:

(P1) any two distinct rows have the same number, λ_{rr}, of common elements;

(P2) any two distinct columns have the same number, λ_{cc}, of common elements.

Such an array is denoted by $DA(v, k, \lambda_{rr}, \lambda_{cc} : r \times c)$. Suppose further that $\mathcal{A}$ satisfies the third property:

(P3) any row and any column have the same number, λ_{rc}, of common elements,

then $\mathcal{A}$ is called a *triple array*, a $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$.

Now consider a pair of distinct elements $x \in V$ and $y \in V$. If both occur in the same row of $\mathcal{A}$ then we say that the pair $\{x, y\}$ occurs in this row, similarly for columns. Suppose that $\{x, y\}$ occurs in r_1 rows of $\mathcal{A}$ and in c_1 columns of $\mathcal{A}$, then we say that it occurs $\mu_{\{x,y\}} = r_1 + c_1$ times in the *grid* $\mathcal{A}$. We call $\mathcal{A}$ a *balanced grid* if there is a constant μ such that $\mu = \mu_{\{x,y\}}$ for every x and y. We denote such a balanced grid by $BG(v, k, \mu : r \times c)$.

In Theorem 6.1 of [3] it was shown that, when $v = r + c - 1$, every triple array $TA(v, k, \lambda_{rr}, \lambda_{cc}, k : r \times c)$ is a balanced grid $BG(v, k, k : r \times c)$. It was then stated that examples to the converse of this Theorem had been found. In Theorem 2.5 of this paper we prove the converse of Theorem 6.1 of [3]. Our main result (Theorem 2.6) is: Let $v = r + c - 1$. Then every triple array is a $TA(v, k, c - k, r - k, k : r \times c)$ and every balanced grid is a $BG(v, k, k : r \times c)$, and they are equivalent.

Finally, we restate a conjecture of Agrawal [1] concerning symmetric balanced incomplete block designs and triple arrays.

2 For v=r+c-1, TA and BG are equivalent

We work mainly with the variables r, c, and k; writing other variables in terms of these three variables, see Theorems 2.2, 3.1, and 4.1 of [3].

$$v = \frac{rc}{k},\ \ \lambda_{rr} = \frac{c(k-1)}{r-1},\ \ \lambda_{cc} = \frac{r(k-1)}{c-1},\ \ \lambda_{rc} = k,\ \ \mu = \frac{k^2(r+c-2)}{rc-k}. \quad (1)$$

When $v = r + c - 1$ if values of the two parameters r and c are given then all parameters in (1) can be expressed in terms of them, and so are 'forced'. But we prefer to keep k in our formulae:

Lemma 2.1

(i) In a triple array $TA(v, k, \lambda_{rr}, \lambda_{cc}, k : r \times c)$ the following are equivalent: $v = r + c - 1$ and $\lambda_{rr} = c - k$ and $\lambda_{cc} = r - k$.

(ii) In a balanced grid $BG(v, k, \mu : r \times c)$ we have $v = r + c - 1$ if and only if $\mu = k$.

Proof. (i) If $v = r + c - 1$ then $c = v - r + 1$. Then $ck = vk - rk + k = rc - rk + k$, and so $ck - c = rc - rk - c + k = (r-1)(c-k)$. But, from (1), $\lambda_{rr} = \frac{c(k-1)}{r-1}$, and so $\lambda_{rr} = c - k$. The converse is given by working backwards. Hence $v = r + c - 1$ if and only if $\lambda_{rr} = c - k$. Similarly we can prove that $v = r + c - 1$ if and only if $\lambda_{cc} = r - k$.
(ii) Suppose that $v = r + c - 1$. Then, from (1), $v = \frac{rc}{k} = r + c - 1$. So $\frac{rc}{k} - 1 = \frac{rc-k}{k} = r + c - 2$. Now (1) gives $\mu = k$. The converse is given by working backwards. ∎

The following Corollary was not explicitly stated in [3].

Corollary 2.2 *When $v = r + c - 1$ every triple array is a $TA(v, k, c - k, r - k, k : r \times c)$, and every balanced grid is a $BG(v, k, k : r \times c)$.* ∎

Matching BIBD's

Let $\mathcal{D}_1$ be a $(v_1, b, r_1, \kappa, \lambda_1) - BIBD$ based on a v_1-set V_1, and $\mathcal{D}_2$ a $(v_2, b, r_2, \kappa, \lambda_2) - BIBD$ based on a v_2-set V_2, with $v_1 v_2 = b\kappa$. Let the b blocks of $\mathcal{D}_1$ be arranged in any fixed order, and let the κ elements in each block be arranged in any fixed order. Then $\mathcal{D}_1$ and $\mathcal{D}_2$ are *matching* if the b blocks of $\mathcal{D}_2$, and the κ elements within each block, can be arranged so that when $\mathcal{D}_2$ is superimposed onto $\mathcal{D}_1$ then each of the $v_1 v_2$ pairs from $V_1 \times V_2$ appears exactly once amongst the $b\kappa$ pairs covered. See Preece [4] Section 6, definition (b), for an equivalent definition of matching $BIBD$'s. Such superimpositions are generally known as Graeco-Latin designs.

Example 1 Two matching $BIBD$'s: a $(5, 10, 6, 3, 3) - BIBD$ based on $\{R_1, R_2, R_3, R_4, R_5\}$ and a $(6, 10, 5, 3, 2) - BIBD$ based on $\{C_1, C_2, C_3, C_4, C_5, C_6\}$, and their superimposition.

R_1	R_2	R_3	C_1	C_4	C_5	R_1C_1	R_2C_4	R_3C_5
R_1	R_3	R_5	C_2	C_3	C_5	R_1C_2	R_3C_3	R_5C_5
R_1	R_3	R_4	C_3	C_5	C_6	R_1C_3	R_3C_6	R_4C_5
R_1	R_4	R_5	C_1	C_3	C_4	R_1C_4	R_4C_3	R_5C_1
R_1	R_2	R_5	C_1	C_5	C_6	R_1C_5	R_2C_1	R_5C_6
R_1	R_2	R_4	C_2	C_4	C_6	R_1C_6	R_2C_2	R_4C_4
R_2	R_4	R_5	C_3	C_4	C_6	R_2C_3	R_4C_6	R_5C_4
R_2	R_3	R_4	C_2	C_4	C_5	R_2C_5	R_3C_4	R_4C_2
R_2	R_3	R_5	C_1	C_2	C_6	R_2C_6	R_3C_1	R_5C_2
R_3	R_4	R_5	C_1	C_2	C_3	R_3C_2	R_4C_1	R_5C_3

Block structures $\mathcal{R}^\perp$, $\mathcal{C}^\perp$, and $\mathcal{S}$

Let $\mathcal{A}$ be an arbitrary array $A(v, k : r \times c)$. Label the r rows of $\mathcal{A}$ with $R_1, R_2, \ldots, R_r$, and the c columns with $C_1, C_2, \ldots, C_c$.

Let $\mathcal{R} = \{R_1, R_2, \ldots, R_r\}$ be the block structure composed of the r rows of $\mathcal{A}$. Similarly, let $\mathcal{C} = \{C_1, C_2, \ldots, C_c\}$ be the block structure composed of the c columns of $\mathcal{A}$.

For any $x \in V$ let $R_x^\perp = \{R_i \,|\, x \in R_i\}$. Then $\mathcal{R}^\perp = \{R_x^\perp \,|\, x \in V\}$ is the dual of $\mathcal{R}$ and is a block structure based on the set $\{R_1, R_2, \ldots, R_r\}$ with v blocks each of size k. Similarly, for any $x \in V$ let $C_x^\perp = \{C_j \,|\, x \in C_j\}$. Then

$\mathcal{C}^\perp = \{C_x^\perp \,|\, x \in V\}$ is the dual of $\mathcal{C}$ and is a block structure based on the set $\{C_1, C_2, \ldots, C_c\}$ with v blocks each of size k.

Define $S_x = R_x^\perp \cup C_x^\perp$ for every $x \in V$, and let $\mathcal{S}$ be the block structure $\{S_x \,|\, x \in V\}$.

By definition of a double array and matching $BIBD$'s we have (compare Lemma 2.1 of [3]):

Lemma 2.3 *Let $\mathcal{A}$ be an arbitrary array $A(v, k : r \times c)$. Then $\mathcal{A}$ is a double array $DA(v, k, \lambda_{rr}, \lambda_{cc} : r \times c)$ if and only if $\mathcal{R}^\perp$ is a $(r, v, c, k, \lambda_{rr}) - BIBD$ and $\mathcal{C}^\perp$ is a $(c, v, r, k, \lambda_{cc}) - BIBD$, and $\mathcal{R}^\perp$ and $\mathcal{C}^\perp$ are matching.* ■

When $\mathcal{A}$ is a double array we call $\mathcal{R}^\perp$ its $BIBD_R$ and $\mathcal{C}^\perp$ its $BIBD_C$.

Example 2 A double array $DA(10, 3, 3, 2 : 5{\times}6)$ whose matching $BIBD_R$ and $BIBD_C$ were given above in Example 1.

	C_1	C_2	C_3	C_4	C_5	C_6
R_1	1	2	3	4	5	6
R_2	5	6	7	1	8	9
R_3	9	10	2	8	1	3
R_4	10	8	4	6	3	7
R_5	4	9	10	7	2	5

Before the next Theorem, we need the following result of Ryser [6], Chapter 8, Theorem 2.2:

Let $\mathcal{B}$ be an incidence structure based on a v-set with v blocks each of size k, in which any two distinct blocks intersect in the same number λ of elements. Then $\mathcal{B}$ is a $(v, k, \lambda) - SBIBD$.

Compare the following Theorem with Theorem 5.2 of [3].

Theorem 2.4 *Let $\mathcal{G}$ be a $BG(v, k, \mu : r \times c)$ with $v = r + c - 1$. Then there exists a $(v + 1, r, r - k)$-SBIBD.*

Proof. Recall the definitions of the block structures $\mathcal{R}^\perp$ and $\mathcal{C}^\perp$ above. Let $B_0 = \{R_1, R_2, \ldots, R_r\}$. For each $x \in V$ put $\overline{R}_x^\perp = B_0 \backslash R_x^\perp$, then $|\overline{R}_x^\perp| = r - k$.

Let $B_x = \overline{R}_x^\perp \cup C_x^\perp$ for each $x \in V$. Then $|B_x| = (r - k) + k = r$. Now consider the block structure $\mathcal{B} = \{B_x | x \in V\} \cup \{B_0\}$. It is based on the $r+c=v+1$ elements from $\mathcal{R}\cup\mathcal{C}=\{R_1,R_2,\ldots,R_r,C_1,C_2,\ldots,C_c\}$ and has $v+1$ blocks each of size r. We now show that $\mathcal{B}$ is the required $(v+1,r,r-k)-SBIBD$.

Now $\mathcal{G}$ is a BG in which every pair $\{x,y\}$ occurs $\mu=k$ (Lemma 2.1(ii)) times, so $|S_x^\perp\cap S_y^\perp|=|R_x^\perp\cap R_y^\perp|+|C_x^\perp\cap C_y^\perp|=k$. We have:

$$\begin{aligned}
|B_x\cap B_y| &= |\overline{R}_x^\perp\cap\overline{R}_y^\perp|+|C_x^\perp\cap C_y^\perp|\\
&= |\overline{R}_x^\perp|+|\overline{R}_y^\perp|-|\overline{R}_x^\perp\cup\overline{R}_y^\perp|+|C_x^\perp\cap C_y^\perp|\\
&= (r-k)+(r-k)-|\overline{R_x^\perp\cap R_y^\perp}|+|C_x^\perp\cap C_y^\perp|\\
&= 2r-2k-(r-|R_x^\perp\cap R_y^\perp|)+|C_x^\perp\cap C_y^\perp|\\
&= r-2k+(|R_x^\perp\cap R_y^\perp|+|C_x^\perp\cap C_y^\perp|)\\
&= r-2k+k=r-k.
\end{aligned}$$

Also, for all $x\in V$, we have $|B_x\cap B_0|=r-k$. Thus any two distinct blocks of $\mathcal{B}$ intersect in $r-k$ elements. So, from Ryser's result above, $\mathcal{B}$ is a $(v+1,r,r-k)-SBIBD$. ■

Next is the converse to Theorem 6.1 of [3]:

Theorem 2.5 *Let $v=r+c-1$. Every $BG(v,k,k:r\times c)$ is a $TA(v,k,c-k,r-k,k:r\times c)$.*

Proof. Let $\mathcal{G}$ be a $BG(v,k,k:r\times c)$. Recall from Theorem 2.4 above that $\mathcal{B}$ is a $(v+1,r,r-k)-SBIBD$. The construction of $\mathcal{B}$ from $\mathcal{R}^\perp$ and $\mathcal{C}^\perp$ gives: Firstly, $\mathcal{R}^\perp$ is the complement of the derived design of $\mathcal{B}$ with respect to block B_0, hence $\mathcal{R}^\perp$ is a $(r,v,c,k,c-k)-BIBD$. Secondly, $\mathcal{C}^\perp$ is the residual design of $\mathcal{B}$ with respect to B_0, hence $\mathcal{C}^\perp$ is a $(c,v,r,k,r-k)-BIBD$. Since $\mathcal{R}^\perp$ and $\mathcal{C}^\perp$ are also constructed from an array, they are matching. Hence, via Lemma 2.3, $\mathcal{G}$ is a double array, a $DA(v,k,c-k,r-k:r\times c)$.

Consider any pair $\{R_i,C_j\}$. Then C_j occurs r times in the first v blocks of $\mathcal{B}$, and pair $\{R_i,C_j\}$ occurs $r-k$ times in these blocks. So, amongst the first v blocks of $\mathcal{B}$, there are $r-(r-k)=k$ blocks which do not contain R_i but do contain C_j. Hence, in $\mathcal{S}$, there are k blocks containing pair $\{R_i,C_j\}$. Thus $|R_i\cap C_j|=k$ for every i and j, and so $\mathcal{G}$ is a triple array, a $TA(v,k,c-k,r-k,k:r\times c)$. ■

Using Theorem 6.1 from [3] and Corollary 2.2 above, we have:

Theorem 2.6 *Let $v = r + c - 1$. Then every triple array is a $TA(v, k, c - k, r - k, k : r \times c)$ and every balanced grid is a $BG(v, k, k : r \times c)$, and they are equivalent.* ∎

Example 3 An array $\mathcal{A}$, a $A(10, 3 : 5 \times 6)$, which is both a balanced grid $BG(10, 3, 3 : 5 \times 6)$ and a triple array $TA(10, 3, 3, 2, 3 : 5 \times 6)$. The three block structures shown are its $BIBD_R$, a $(5, 10, 6, 3, 3) - BIBD$; its $BIBD_C$, a $(6, 10, 5, 3, 2) - BIBD$; and $\mathcal{B}$, a $(11, 5, 2) - SBIBD$.

	C_1	C_2	C_3	C_4	C_5	C_6
R_1	1	2	3	4	5	6
R_2	4	7	1	3	8	9
R_3	2	5	10	8	9	3
R_4	10	8	7	6	1	2
R_5	9	4	5	10	6	7

R_1	R_2	R_4	C_1	C_3	C_5	R_3	R_5	C_1	C_3	C_5
R_1	R_3	R_4	C_1	C_2	C_6	R_2	R_5	C_1	C_2	C_6
R_1	R_2	R_3	C_3	C_4	C_6	R_4	R_5	C_3	C_4	C_6
R_1	R_2	R_5	C_1	C_2	C_4	R_3	R_4	C_1	C_2	C_4
R_1	R_3	R_5	C_2	C_3	C_5	R_2	R_4	C_2	C_3	C_5
R_1	R_4	R_5	C_4	C_5	C_6	R_2	R_3	C_4	C_5	C_6
R_2	R_4	R_5	C_2	C_3	C_6	R_1	R_3	C_2	C_3	C_6
R_2	R_3	R_4	C_2	C_4	C_5	R_1	R_5	C_2	C_4	C_5
R_2	R_3	R_5	C_1	C_5	C_6	R_1	R_4	C_1	C_5	C_6
R_3	R_4	R_5	C_1	C_3	C_4	R_1	R_2	C_1	C_3	C_4
						R_1	R_2	R_3	R_4	R_5

Agrawal's Conjecture

The second paragraph in the proof of Theorem 2.5 above is essentially Agrawal's method of constructing a triple array $TA(v, k, c-k, r-k, k : r \times c)$ with $v = r+c-1$ from a $(v+1, r, r-k) - SBIBD$ with $k > 2$, see Agrawal [1]. It seems worthwhile to restate his conjecture in terms of matching BIBD's:

Let $\mathcal{S}$ be a $(v_s, k_s, \lambda_s) - SBIBD$ with $k_s - \lambda_s > 2$. For any fixed block S_0 let $\mathcal{S}_{\text{der}}$ denote the derived design of $\mathcal{S}$ with respect to S_0, and let $\mathcal{S}_{\text{res}}$ denote the residual design of $\mathcal{S}$ with respect to S_0.

Then the complement of $\mathcal{S}_{\text{der}}$ and $\mathcal{S}_{\text{res}}$ are matching.

An incorrect proof of this conjecture appeared in Raghavarao and Nageswararao [5], as was pointed out in Bailey and Heidtmann [2], and Wallis and Yucas [7]. It appears that this conjecture is still open.

If Agrawal's conjecture is correct then any $(v_s, k_s, \lambda_s) - SBIBD$ with $k_s - \lambda_s > 2$ gives rise to a $TA(v_s - 1, k_s - \lambda_s, v_s - 2k_s + \lambda_s, \lambda_s, k_s - \lambda_s : k_s \times (v_s - k_s))$, a triple array with '$v = r + c - 1$'.

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