Hindawi Publishing Corporation
International Journal of Combinatorics
Volume 2011, Article ID 539030, 29 pages
doi:10.1155/2011/539030

Research Article

Zeons, Permanents, the Johnson Scheme, and Generalized Derangements

Philip Feinsilver and John McSorley

Department of Mathematics, Southern Illinois University, Carbondale, IL 62901, USA

Correspondence should be addressed to Philip Feinsilver, pfeinsil@math.siu.edu

Received 20 January 2011; Accepted 1 April 2011

Academic Editor: Alois Panholzer

Starting with the zero-square "zeon algebra," the connection with permanents is shown. Permanents of submatrices of a linear combination of the identity matrix and all-ones matrix lead to moment polynomials with respect to the exponential distribution. A permanent trace formula analogous to MacMahon's master theorem is presented and applied. Connections with permutation groups acting on sets and the Johnson association scheme arise. The families of numbers appearing as matrix entries turn out to be related to interesting variations on derangements. These generalized derangements are considered in detail as an illustration of the theory.

1. Introduction

Functions acting on a finite set can be conveniently expressed using matrices, whereby the composition of functions corresponds to multiplication of the matrices. Essentially, one is considering the induced action on the vector space with the elements of the set acting as a basis. This action extends to tensor powers of the vector space. One can take symmetric powers, antisymmetric powers, and so forth, that yield representations of the multiplicative semigroup of functions. An especially interesting representation occurs by taking nonreflexive, symmetric powers. Identifying the underlying set of cardinality n with $\{1, 2, \ldots, n\}$, the vector space has basis $e_1, e_2, \ldots$. The action we are interested in may be found by saying that the elements e_i generate a "zeon algebra," the relations being that the e_i commute, with $e_i^2 = 0$, $1 \le i \le n$. To get a feeling for this, first we recall the action on Grassmann algebra where the matrix elements of the induced action arise as determinants. For the zeon case, permanents appear.

An interesting connection with the centralizer algebra of the action of the symmetric group comes up. For the defining action on the set $\{1, \ldots, n\}$, represented as 0-1 permutation matrices, the centralizer algebra of $n \times n$ matrices commuting with the entire group is generated by I, the identity matrix, and J, the all-ones matrix. The question was if they would help determine the centralizer algebra for the action on subsets of a fixed size, ℓ-sets, for $\ell > 1$. It is known that the basis for the centralizer algebra is given by the adjacency matrices of the Johnson scheme. Could one find this working solely with I and J? The result is that by computing the "zeon powers", that is, the action of $sI + tJ$, linear combinations of I and J, on ℓ-sets, the Johnson scheme appears naturally. The coefficients are polynomials in s and t occurring as moments of the exponential distribution. And they turn out to count derangements and related generalized derangements. The occurrence of Laguerre polynomials in the combinatorics of derangements is well known. Here, the ${}_2F_1$ hypergeometric function, which is closely related to Poisson-Charlier polynomials, arises rather naturally.

Here is an outline of the paper. Section 2 introduces zeons and permanents. The trace formula is proved. Connections with the centralizer algebra of the action of the symmetric group on sets are detailed. Section 3 is a study of exponential polynomials needed for the remainder of the paper. Zeon powers of $sI+tJ$ are found in Section 4 where the spectra of the matrices are found via the Johnson scheme. Section 5 presents a combinatorial approach to the zeon powers of $sI+tJ$, including an interpretation of exponential moment polynomials by elementary subgraphs. In Section 6, generalized derangement numbers, specifically counting derangements and counting arrangements, are considered in detail. The Appendix has some derangement numbers and arrangement numbers for reference, as well as a page of exponential polynomials. An example expressing exponential polynomials in terms of elementary subgraphs is given there.

2. Representations of Functions Acting on Sets

Let $\mathcal{V}$ denote the vector space $\mathbb{Q}^n$ or $\mathbb{R}^n$. We will look at the action of a linear map on $\mathcal{V}$ extended to quotients of tensor powers $\mathcal{V}^{\otimes \ell}$. We work with coordinates rather than vectors. First, recall the Grassmann case. To find the action on $\mathcal{V}^{\wedge \ell}$ consider an algebra generated by n variables e_i satisfying $e_i e_j = -e_j e_i$. In particular, $e_i^2 = 0$.

Notation. The standard n-set $\{1, \ldots, n\}$ will be denoted $[n]$. Roman caps I, J, A, and so forth denote subsets of $[n]$. We will identify them with the corresponding ordered tuples. Generally, given an n-tuple $(x_1, \ldots, x_n)$ and a subset $\mathrm{I} \subset [n]$, we denote products

$$x_{\mathrm{I}} = \prod_{j \in \mathrm{I}} x_j, \tag{2.1}$$

where the indices are in increasing order if the variables are not assumed to commute.

As an index, we will use U to denote the full set $[n]$.

Italic I and J will denote the identity matrix and all-ones matrix, respectively.

For a matrix X_{IJ}, say, where the labels are subsets of fixed size l, dictionary ordering is used. That is, convert to ordered tuples and use dictionary ordering. For example, for $n = 4$, $l = 2$, we have labels 12, 13, 14, 23, 24, and 34 for rows one through six, respectively.

A basis for $\mathcal{V}^{\wedge\ell}$ is given by products

$$e_{\mathrm{I}} = e_{i_1} e_{i_2} \cdots e_{i_\ell}, \tag{2.2}$$

$\mathrm{I} \subset [n]$, where we consider I as an ordered ℓ-tuple. Given a matrix X acting on $\mathcal{V}$, let

$$y_i = \sum_j X_{ij} e_j, \tag{2.3}$$

with corresponding products y_{I}, then the matrix $X^{\wedge\ell}$ has entries given by the coefficients in the expansion

$$y_{\mathrm{I}} = \sum_{\mathrm{J}} \left(X^{\wedge\ell}\right)_{\mathrm{IJ} e_{\mathrm{J}}} \tag{2.4}$$

where the anticommutation rules are used to order the factors in e_{J}. Note that the coefficient of e_j in y_i is X_{ij} itself. And for $n > 3$, the coefficient of e_{34} in y_{12} is

$$\det\begin{pmatrix} X_{13} & X_{14} \\ X_{23} & X_{24} \end{pmatrix}. \tag{2.5}$$

We see that in general the IJ entry of $X^{\wedge\ell}$ is the minor of X with row labels I and column labels J. A standard term for the matrix $X^{\wedge\ell}$ is a *compound matrix*. Noting that $X^{\wedge\ell}$ is $\binom{n}{\ell} \times \binom{n}{\ell}$, in particular, $\ell = n$ yields the one-by-one matrix with entry equal to $(X^{\wedge n})_{\mathrm{UU}} = \det X$.

In this work, we will use the algebra of *zeons*, standing for "zero-ons" , or more specifically, "zero-square bosons". That is, we assume that the variables e_i satisfy the properties

$$e_i e_j = e_j e_i, \qquad e_i^2 = 0. \tag{2.6}$$

A basis for the algebra is again given by e_{I}, $\mathrm{I} \subset [n]$. At level ℓ, the induced matrix $X^{\vee\ell}$ has IJ entries according to the expansion of y_{I},

$$y_i = \sum_j X_{ij} e_j \longrightarrow y_I = \sum_J \left(X^{\vee\ell}\right)_{\mathrm{IJ} e_{\mathrm{J}}} \tag{2.7}$$

similar to the Grassmann case. Since the variables commute, we see that the IJ entry of $X^{\vee\ell}$ is the *permanent* of the submatrix with rows I and columns J. In particular, $(X^{\vee n})_{\mathrm{UU}} = \operatorname{per} X$. We refer to the matrix $X^{\vee\ell}$ as the "ℓth zeon power of X."

2.1. Functions on the Power Set of $[n]$

Note that $X^{\vee\ell}$ is indexed by ℓ-sets. Suppose that X_f represents a function $f : [n] \to [n]$. So it is a zero-one matrix with $(X_f)_{ij} = 1$, the single entry in row i if f maps i to j.

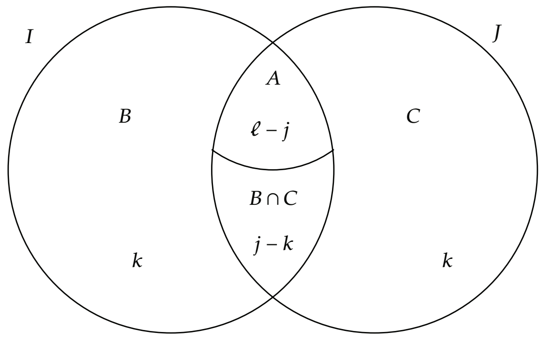

Figure 1: Configuration of sets: B = I \ A, C = J \ A.

The ℓth zeon power of X is the matrix of the induced map on ℓ-sets. If f maps an ℓ-set I to one of lower cardinality, then the corresponding row in $X^{\vee\ell}$ has all zero entries. Thus, the induced matrices in general correspond to "partial functions".

However, if X is a permutation matrix, then $X^{\vee\ell}$ is a permutation matrix for all $0 \le \ell \le n$. So, given a group of permutation matrices, the map $X \to X^{\vee\ell}$ is a representation of the group.

2.2. Zeon Powers of $sI + tX$

Our main theorem computes the ℓth zeon power of $sI + tX$ for an $n \times n$ matrix X, where s and t are scalar variables. Figure 1 illustrates the proof.

Theorem 2.1. *For a given matrix* X, *for* $0 \le \ell \le n$, *and indices* $|\mathrm{I}| = |\mathrm{J}| = \ell$,

$$\left((sI + t\mathrm{X})^{\vee\ell}\right)_{\mathrm{IJ}} = \sum_{0 \le j \le \ell} s^{\ell-j} t^{j} \sum_{\substack{A \subset \mathrm{I} \cap \mathrm{J} \\ |\mathrm{A}| = \ell - j}} \left(\mathrm{X}^{\vee j}\right)_{I \setminus A, J \setminus A}. \tag{2.8}$$

Proof. Start with $y_i = se_i + t\xi_i$, where $\xi_i = \sum_j \mathrm{X}_{ij} e_j$. Given $\mathrm{I} = (i_1, \ldots, i_\ell)$, we want the coefficient of e_{J} in the expansion of the product $y_I = y_{i_1} \cdots y_{i_\ell}$. Now,

$$y_I = (se_{i_1} + t\xi_{i_1}) \cdots (se_{i_\ell} + t\xi_{i_\ell}). \tag{2.9}$$

Choose $\mathrm{A} \subset \mathrm{I}$ with $|\mathrm{A}| = \ell - j$, $0 \le j \le \ell$. A typical term of the product has the form

$$s^{\ell-j} t^{j} e_{\mathrm{A}} \xi_{\mathrm{B}} \tag{2.10}$$

where $\mathrm{A} \cap \mathrm{B} = \emptyset$, $\mathrm{B} = \mathrm{I} \setminus \mathrm{A}$. ξ_{B} denotes the product of terms ξ_{i} with indices in B. Expanding, we have

$$\begin{aligned} \xi_{\mathrm{B}} &= \sum_{C} \left(X^{\vee j}\right)_{\mathrm{BC}} e_{\mathrm{C}}, \\ \mathrm{e}_{A} \xi_{\mathrm{B}} &= \sum_{C} \left(X^{\vee j}\right)_{\mathrm{BC}} e_{\mathrm{A}} e_{\mathrm{C}}. \end{aligned} \tag{2.11}$$

Thus, for a contribution to the coefficient of e_J, we have $A \cup C = J$, where $A \cap C = \emptyset$. That is, $C = J \setminus A$ and $A \subset I \cap J$. So, the coefficient of $s^{\ell-j}t^j$ is as stated. □

2.3. Trace Formula

Another main feature is the *trace formula* which shows the permanent of $I + tX$ as the generating function for the traces of the zeon powers of X. This is the zeon analog of the theorem of MacMahon for representations on symmetric tensors.

Theorem 2.2. *One has the formula*

$$\operatorname{per}(sI + tX) = \sum_j s^{n-j}t^j \operatorname{tr} X^{\vee j}. \tag{2.12}$$

Proof. The permanent of $sI + tX$ is the UU entry of $(sI + tX)^{\vee n}$. Specialize $I = J = U$ in Theorem 2.1. So A is any $(n-j)$-set with $I \setminus A = J \setminus A = A'$, its complement in $[n]$. Thus,

$$\begin{aligned}
\operatorname{per}(sI + tX) &= \left((sI + tX)^{\vee n}\right)_{UU} \\
&= \sum_{0 \le j \le n} s^{n-j}t^j \sum_{|A|=n-j} \left(X^{\vee j}\right)_{A',A'} \\
&= \sum_{0 \le j \le n} s^{n-j}t^j \operatorname{tr} X^{\vee j},
\end{aligned} \tag{2.13}$$

as required. □

2.4. Permutation Groups

Let X be an $n \times n$ permutation matrix. We can express per $(I + tX)$ in terms of the cycle decomposition of the associated permutation.

Proposition 2.3. *For a permutation matrix X,*

$$\operatorname{per}(I + tX) = \prod_{0 \le \ell \le n} \left(1 + t^{\ell}\right)^{n_X(\ell)}, \tag{2.14}$$

where $n_X(\ell)$ is the number of cycles of length ℓ in the cycle decomposition of the corresponding permutation.

Proof. Decomposing the permutation associated to X yields a decomposition into invariant subspaces of the underlying vector space $\mathcal{V}$. So per $(I + tX)$ will be the product of per $(I + tX_c)$ as c runs through the corresponding cycles with X_c the restriction of X to the invariant subspace for each c. So we have to check that if X acts on $\mathcal{V}^{\ell}$ as a cycle of length ℓ, then per $(I + tX) = 1 + t^{\ell}$. For this, apply Theorem 2.2. Apart from level zero, there is only one set fixed by any $X^{\vee j}$, namely when $j = \ell$. So the trace of $X^{\vee j}$ is zero unless $j = \ell$ and then it is one. The result follows. □

2.4.1. Cycle Index: Orbits on ℓ-sets

Now, consider a group, G, of permutation matrices. We have the cycle index

$$Z_G(z_1, z_2, \ldots, z_n) = \frac{1}{|G|} \sum_{X \in G} z_1^{n_X(1)} z_2^{n_X(2)} \cdots z_n^{n_X(n)}, \tag{2.15}$$

each z_ℓ corresponding to ℓ-cycles in the cycle decomposition associated to the Xs. From Proposition 2.3, we have an expression in terms of permanents. Combining with the trace formula, we get the following.

Theorem 2.4. *Let G be a permutation group of matrices, then one has*

$$\begin{aligned} \frac{1}{|G|} \sum_{X \in G} \operatorname{per}(I + tX) &= Z_G\left(1 + t, 1 + t^2, \ldots, 1 + t^n\right) \\ &= \sum_{\ell} t^{\ell} \ \#(\text{orbits on } l\text{-sets}). \end{aligned} \tag{2.16}$$

Remark 2.5. This result refers to three essential theorems in group theory acting on sets. Equality of the first and last expressions is the "permanent" analog of Molien's theorem, which is the case for a group acting on the symmetric tensor algebra, that the cycle index counts orbits on subsets is an instance of Polya Counting, with two colors. The last expression is followed by the Cauchy-Burnside lemma applied to the groups $G^{\vee \ell} = \{X^{\vee \ell}\}_{X \in G}$.

2.4.2. Centralizer Algebra and Johnson Scheme

Given a group, G, of permutation matrices, an important question is to determine the set (among all matrices) of matrices commuting with all of the matrices in G. This is the *centralizer algebra* of the group. For the symmetric group, the only matrices are I and J. For the action of the symmetric group on ℓ-sets, a basis for the centralizer algebra is given by the incidence matrices for the Johnson distance. These are the same as the adjacency matrices for the Johnson (association) scheme. Recall that the Johnson distance between two ℓ-sets I and J is

$$\operatorname{dist}_{\mathrm{JS}}(\mathrm{I}, \mathrm{J}) = \frac{1}{2} |\mathrm{I} \ \Delta \ \mathrm{J}| = |\mathrm{I} \setminus \mathrm{J}| = |\mathrm{J} \setminus \mathrm{I}|. \tag{2.17}$$

The corresponding matrices $\mathrm{JS}_k^{n\ell}$ are defined by

$$\left(\mathrm{JS}_k^{n\ell}\right)_{\mathrm{IJ}} = \begin{cases} 1 & \text{if } \operatorname{dist}_{\mathrm{JS}}(\mathrm{I}, \mathrm{J}) = k, \\ 0, & \text{otherwise}. \end{cases} \tag{2.18}$$

As it is known, [1, page 36], that a basis for the centralizer algebra is given by the orbits of the group G^2, acting on pairs, the Johnson basis is a basis for the centralizer algebra. Since the Johnson distance is symmetric, it suffices to look at $G^{\vee 2}$.

Now, we come to the question that is a starting point for this work. If I and J are the only matrices commuting with all elements (as matrices) of the symmetric group, then since the map $G \to G^{\vee\ell}$ is a homomorphism, we know that $I^{\vee\ell}$ and $J^{\vee\ell}$ are in the centralizer algebra of $G^{\vee\ell}$. The question is how to obtain the rest? The, perhaps surprising, answer is that in fact one can obtain the complete Johnson basis from I and J alone. This will be one of the main results, Theorem 4.1.

2.4.3. Permanent of $sI + tJ$

First, let us consider $sI + tJ$.

Proposition 2.6. *One has the formula*

$$\operatorname{per}(sI + tJ) = n! \sum_{0 \le \ell \le n} \frac{s^{\ell} t^{n-\ell}}{\ell!}. \tag{2.19}$$

Proof. For $X = J$, we see directly, since all entries equal one in all submatrices, that

$$\left(J^{\vee\ell}\right)_{\mathrm{IJ}} = \ell! \tag{2.20}$$

for all I and J. Taking traces,

$$\operatorname{tr} J^{\vee\ell} = \binom{n}{\ell} \ell!, \tag{2.21}$$

and by the trace formula, Theorem 2.2,

$$\operatorname{per}(sI + tJ) = \sum_{\ell} \binom{n}{\ell} \ell!\, s^{n-\ell} t^{\ell} = \sum_{\ell} \frac{n!}{(n-\ell)!} s^{n-\ell} t^{\ell}. \tag{2.22}$$

Reversing the order of summation yields the result stated. □

Corollary 2.7. *For varying n, one will explicitly denote $p_n(s,t) = \operatorname{per}(sI_n + tJ_n)$, then, with $p_0(s,t) = 1$,*

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} p_n(s,t) = \frac{e^{sz}}{1 - tz}. \tag{2.23}$$

The Corollary exhibits the operational formula

$$p_n(s,t) = \frac{1}{1 - tD_s} s^n, \tag{2.24}$$

where $D_s = d/ds$. By inspection, this agrees with (2.19) as well.

Observe that (2.19) can be rewritten as

$$\operatorname{per}(sI + tJ) = \int_0^\infty (s + ty)^n e^{-y} dy, \tag{2.25}$$

that is, these are "moment polynomials" for the exponential distribution with an additional scale parameter.

We proceed to examine these moment polynomials in detail.

3. Exponential Polynomials

For the exponential distribution, with density e^{-y} on $(0, \infty)$, the *moment polynomials* are defined as

$$h_n(x) = \int_0^\infty (x + y)^n e^{-y} dy. \tag{3.1}$$

The exponential embeds naturally into the family of weights of the form $x^m\, e^{-x}$ on $(0, \infty)$ as for generalized Laguerre polynomials. We define correspondingly

$$h_{n,m}(x, t) = \int_0^\infty (x + ty)^n (ty)^m e^{-y} dy, \tag{3.2}$$

for nonnegative integers n, m, introducing a factor of y^m and a scale factor of t. We refer to these as *exponential moment polynomials*.

Proposition 3.1. *Observe the following properties of the exponential moment polynomials.*

(1) *The generating function*

$$\frac{1}{t^m m!} \sum_{n=0}^{\infty} \frac{z^n}{n!} h_{n,m}(x, t) = \frac{e^{zx}}{(1 - tz)^{1+m}}, \tag{3.3}$$

for $|tz| < 1$.

(2) *The operational formula*

$$\frac{1}{t^m m!}\, h_{n,m}(x, t) = (I - tD)^{-(m+1)} x^n, \tag{3.4}$$

where I *is the identity operator and* $D = d/dx$.

(3) *The explicit form*

$$h_{n,m}(x, t) = \sum_{j=0}^{n} \binom{n}{j} (m + j)! x^{n-j} t^{m+j}. \tag{3.5}$$

Proof. For the first formula, multiply the integral by $z^n/n!$ and sum to get

$$\int_0^\infty y^m e^{zx+zty-y}dy = e^{zx}\int_0^\infty y^m e^{-y(1-tz)}dy, \tag{3.6}$$

which yields the stated result.

For the second, write

$$t^m m!(I-tD)^{-(m+1)}x^n = t^m\int_0^\infty y^m e^{-(I-tD)y}x^n dy = \int_0^\infty (ty)^m e^{-y}(x+ty)^n dy, \tag{3.7}$$

using the shift formula $e^{aD}f(x) = f(x+a)$.

For the third, expand $(x+ty)^n$ by the binomial theorem and integrate. □

A variation we will encounter in the following is

$$h_{n-m,m}(x,t) = \sum_{j=0}^{n-m}\binom{n-m}{j}(m+j)!x^{n-m-j}t^{m+j} \tag{3.8}$$

$$= \sum_{j=m}^{n}\binom{n-m}{j-m}j!x^{n-j}t^j \tag{3.9}$$

$$= \sum_{j=m}^{n}\binom{n-m}{n-j}j!x^{n-j}t^j \tag{3.10}$$

$$= \sum_{j=0}^{n-m}\binom{n-m}{j}(n-j)!x^j t^{n-j}, \tag{3.11}$$

replacing the index $j \leftarrow j-m$ for (3.9) and reversing the order of summation for the last line. And for future reference, consider the integral formula

$$h_{n-m,m}(x,t) = \int_0^\infty (x+ty)^{n-m}(ty)^m e^{-y}dy. \tag{3.12}$$

3.1. Hypergeometric Form

Generalized hypergeometric functions provide expressions for the exponential moment polynomials that are often convenient. In the present context, we will use ${}_2F_0$ functions, defined by

$${}_2F_0\left(\begin{matrix} a,b \\ - \end{matrix}\,\middle|\, x\right) = \sum_{j=0}^\infty \frac{(a)_j(b)_j}{j!}x^j, \tag{3.13}$$

where $(a)_j = \Gamma(a+j)/\Gamma(a)$ is the usual Pochhammer symbol. In particular, if a, for example, is a negative integer, the series reduces to a polynomial. Rearranging factors in the expressions for $h_{n,m}$, via (3) in Proposition 3.1, and $h_{n-m,m}$, (3.8), we can formulate these as ${}_2F_0$ hypergeometric functions.

Proposition 3.2. *One has the following expressions for exponential moment polynomials:*

$$\begin{aligned} h_{n,m}(x,t) &= x^n t^m m!\ {}_2F_0\left(\begin{matrix} -n, 1+m \\ — \end{matrix} \,\middle|\, -\frac{t}{x} \right), \\ h_{n-m,m}(x,t) &= x^{n-m} t^m m!\ {}_2F_0\left(\begin{matrix} m-n, 1+m \\ — \end{matrix} \,\middle|\, -\frac{t}{x} \right). \end{aligned} \tag{3.14}$$

4. Zeon Powers of $sI + tJ$

We want to calculate $(sI+tJ)^{\vee \ell}$, that is, the $\binom{n}{\ell} \times \binom{n}{\ell}$ matrix with rows and columns labelled by ℓ-subsets $\mathrm{I}, \mathrm{J} \subset \{1, \ldots, n\}$ with the IJ entry equal to the permanent of the corresponding submatrix of $sI + tJ$. This is equivalent to the induced action of the original matrix $sI + tJ$ on the ℓth zeon space $\mathcal{V}^{\vee \ell}$.

Theorem 4.1. *The ℓth zeon power of $sI + tJ$ is given by*

$$(sI+tJ)^{\vee \ell} = \sum_k \sum_{j=k}^{\ell} \binom{\ell-k}{\ell-j} j! s^{\ell-j} t^j \mathrm{JS}_k^{n\ell} = \sum_k h_{\ell-k,k}(s,t) \mathrm{JS}_k^{n\ell}, \tag{4.1}$$

where the h's are exponential moment polynomials.

Proof. Choose I and J with $|\mathrm{I}| = |\mathrm{J}| = \ell$. By Theorem 2.1, we have, using the fact that all of the entries of $J^{\vee j}$ are equal to $j!$,

$$\begin{aligned} \left((sI+tJ)^{\vee \ell}\right)_{\mathrm{IJ}} &= \sum_{0 \le j \le \ell} s^{\ell-j} t^j \sum_{\substack{A \subset \mathrm{I} \cap \mathrm{J} \\ |\mathrm{A}| = \ell - j}} \left(J^{\vee j}\right)_{\mathrm{I} \setminus \mathrm{A}, \mathrm{J} \setminus \mathrm{A}} \\ &= \sum_{0 \le j \le \ell} s^{\ell-j} t^j \sum_{\substack{\mathrm{A} \subset \mathrm{I} \cap \mathrm{J} \\ |\mathrm{A}| = \ell - \mathrm{j}}} j!\,. \end{aligned} \tag{4.2}$$

Now, if $\mathrm{dist}_{\mathrm{JS}}(\mathrm{I},\mathrm{J}) = k$, then $|\mathrm{I} \cap \mathrm{J}| = \ell - k$, and there are $\binom{\ell-k}{\ell-j}$ subsets A of $\mathrm{I} \cap \mathrm{J}$ satisfying the conditions of the sum. Hence the result. □

Note that the specialization $\ell = n$, $k = 0$, recovers (2.19).

We can write the above expansion using the hypergeometric form of the exponential moment polynomials, Proposition 3.2,

$$(sI+tJ)^{\vee \ell} = \sum_k s^{\ell-k} t^k k!\ {}_2F_0\left(\begin{matrix} k-\ell, 1+k \\ — \end{matrix} \,\middle|\, -\frac{t}{s} \right) \mathrm{JS}_k^{n\ell}. \tag{4.3}$$

4.1. Spectrum of the Johnson Matrices

Recall, for example, [2, page 220], that the spectrum of the Johnson matrices for given n and ℓ is the set of numbers

$$\Lambda_k^{n\ell}(\alpha) = \sum_i \binom{\ell-\alpha}{i}\binom{n-\ell-\alpha+i}{i}\binom{\ell-i}{k-i}(-1)^{k-i}, \tag{4.4}$$

where the eigenvalue for given α has multiplicity $\binom{n}{\alpha} - \binom{n}{\alpha-1}$.

For ℓ-sets, the Johnson distance takes values from 0 to $\min(\ell, n-\ell)$, with α taking values from that same range.

4.2. The Spectrum of $(sI + tJ)^{\vee\ell}$

Recall that as the Johnson matrices are symmetric and generate a commutative algebra, they are simultaneously diagonalizable by an orthogonal transformation of the underlying vector space. Diagonalizing the equation in Theorem 4.1, we see that the spectrum of $(sI + tJ)^{\vee\ell}$ is given by

$$\sum_k h_{\ell-k,k}(s,t)\Lambda_k^{n\ell}(\alpha). \tag{4.5}$$

Proposition 4.2. *The spectrum of $(sI + tJ)^{\vee\ell}$ is given by*

$$\frac{s^\alpha}{t^{n-\ell-\alpha}(n-\ell-\alpha)!}h_{\ell-\alpha,n-\ell-\alpha}(s,t) = \sum_i s^{\ell-i}t^i\binom{\ell-\alpha}{i}\binom{n-\ell-\alpha+i}{i}i!, \tag{4.6}$$

for $0 \le \alpha \le \min(\ell, n-\ell)$, with respective multiplicities $\binom{n}{\alpha} - \binom{n}{\alpha-1}$.

Proof. In the sum over i in (4.4), only the last two factors involve k. We have

$$\begin{aligned}
&\sum_k h_{\ell-k,k}(s,t)(-1)^{k-i}\binom{\ell-i}{k-i} \\
&\quad= \sum_k \int_0^\infty (s+ty)^{\ell-k}(ty)^k(-1)^{k-i}\binom{\ell-i}{k-i}e^{-y}\,dy \quad \text{setting } k = i+m \\
&\quad= \sum_m \int_0^\infty (s+ty)^{\ell-i-m}(ty)^{i+m}(-1)^m\binom{\ell-i}{m}e^{-y}\,dy \\
&\quad= \int_0^\infty (s+ty-ty)^{\ell-i}(ty)^i e^{-y}\,dy \\
&\quad= s^{\ell-i}t^i i!,
\end{aligned} \tag{4.7}$$

using the binomial theorem to sum out m. Filling in the additional factors yields

$$\sum_k h_{\ell-k,k}(s,t)\ \Lambda_k^{n\ell}(\alpha) \;=\sum_i s^{\ell-i}t^i i!\binom{\ell-\alpha}{i}\binom{n-\ell-\alpha+i}{i}, \tag{4.8}$$

Taking out a denominator factor of $(n-\ell-\alpha)!$ and multiplying by $s^{-\alpha}t^{n-\ell-\alpha}$ gives

$$\sum_i s^{\ell-\alpha-i}t^{n-\ell-\alpha+i}\binom{\ell-\alpha}{i}(n-\ell-\alpha+i)!, \tag{4.9}$$

which is precisely $h_{\ell-\alpha,n-\ell-\alpha}$ as in the third statement of Proposition 3.1. □

As in Proposition 3.2, we can express the eigenvalues as follows.

Corollary 4.3. *The spectrum of $(sI+tJ)^{\vee\ell}$ consists of the eigenvalues*

$$s^\ell\,{}_2F_0\left(\begin{matrix}\alpha-\ell,1+n-\ell-\alpha\\ —\end{matrix}\,\middle|\,-\frac{t}{s}\right), \tag{4.10}$$

for $0\le\alpha\le\min(\ell,n-\ell)$, with corresponding multiplicities as indicated above.

4.3. Row Sums and Trace Identity

For the row sums, we know that the all-ones vector is a common eigenvector of the Johnson basis corresponding to $\alpha=0$. These are the valencies $\Lambda_k(0)$. For the Johnson scheme, we have

$$\Lambda_k^{n\ell}(0)=\binom{\ell}{k}\binom{n-\ell}{k}. \tag{4.11}$$

for example, see [2, page 219], which can be checked directly from the formula for $\Lambda_k^{n\ell}(\alpha)$, (4.4), with α set to zero. Setting $\alpha=0$ in Proposition 4.2 gives

$$\frac{1}{t^{n-\ell}\ (n-\ell)!}h_{\ell,n-\ell}(s,t)=\sum_i\binom{\ell}{i}\binom{n-\ell+i}{i}i!s^{\ell-i}t^i, \tag{4.12}$$

for the row sums of $(sI+tJ)^{\vee\ell}$.

4.3.1. Trace Identity

Terms on the diagonal are the coefficient of $\mathrm{JS}_0^{n\ell}$, which is the identity matrix. So, the trace is

$$\operatorname{tr}(sI+tJ)^{\vee\ell}=\binom{n}{\ell}h_{\ell,0}(s,t)=\binom{n}{\ell}\sum_k\binom{\ell}{k}k!s^{\ell-k}t^k. \tag{4.13}$$

Cancelling factorials and reversing the order of summation on k yields the following formula.

$$\operatorname{tr}(sI+tJ)^{\vee\ell} = \frac{n!}{(n-\ell)!}\sum_{0\le k\le\ell}\frac{s^k t^{\ell-k}}{k!}. \tag{4.14}$$

Now, Proposition 4.2 gives the trace

$$\operatorname{tr}\,(sI+tJ)^{\vee\ell} = \sum_{0\le\alpha\le\min(\ell,n-\ell)}\left[\binom{n}{\alpha}-\binom{n}{\alpha-1}\right]\sum_i s^{\ell-i}t^i\binom{\ell-\alpha}{i}\binom{n-\ell-\alpha+i}{i}i!. \tag{4.15}$$

Proposition 4.4. *Equating the above expressions for the trace yields the identity*

$$\sum_{0\le\alpha\le\min(\ell,n-\ell)}\left[\binom{n}{\alpha}-\binom{n}{\alpha-1}\right]\sum_i s^{\ell-i}t^i\binom{\ell-\alpha}{i}\binom{n-\ell-\alpha+i}{i}i! = \frac{n!}{(n-\ell)!}\sum_{0\le j\le\ell}\frac{s^j t^{\ell-j}}{j!}. \tag{4.16}$$

Example 4.5. For $n=4$, $\ell=2$, we have

$$\begin{bmatrix} s^2+2st+2t^2 & st+2t^2 & st+2t^2 & st+2t^2 & st+2t^2 & 2t^2 \\ st+2t^2 & s^2+2st+2t^2 & st+2t^2 & st+2t^2 & 2t^2 & st+2t^2 \\ st+2t^2 & st+2t^2 & s^2+2st+2t^2 & 2t^2 & st+2t^2 & st+2t^2 \\ st+2t^2 & st+2t^2 & 2t^2 & s^2+2st+2t^2 & st+2t^2 & st+2t^2 \\ st+2t^2 & 2t^2 & st+2t^2 & st+2t^2 & s^2+2st+2t^2 & st+2t^2 \\ 2t^2 & st+2t^2 & st+2t^2 & st+2t^2 & st+2t^2 & s^2+2st+2t^2 \end{bmatrix}. \tag{4.17}$$

One can check that the entries are in agreement with Theorem 4.1. The trace is $6s^2+12st+12t^2$. The spectrum is

$$\begin{gathered} \text{eigenvalue } s^2+6st+12t^2, \quad \text{with multiplicity 1,} \\ \text{eigenvalue } s^2+2st, \quad \text{with multiplicity 3,} \\ \text{eigenvalue } s^2, \quad \text{with multiplicity 2,} \end{gathered} \tag{4.18}$$

and the trace can be verified from these as well.

Remark 4.6. What is interesting is that these matrices have polynomial entries with all eigenvalues polynomials as well, and furthermore, the exact same set of polynomials produces the eigenvalues as well as the entries. Specializing s and t to integers, a similar statement holds. All of these matrices will have integer entries with integer eigenvalues, all of which belong to closely related families of numbers. We will examine interesting cases of this phenomenon later on in this paper.

5. Permanents from $sI + tJ$

Here, we present a proof via recursion of the subpermanents of $sI + tJ$, thereby recovering Theorem 4.1 from a different perspective.

Remark 5.1. For the remainder of this paper, we will work with an $n \times n$ matrix corresponding to an $\ell \times \ell$ submatrix of the above discussion. Here, we have blown up the submatrix to full size as the object of consideration.

Let $M_{n,\ell}$ denote the $n \times n$ matrix with $n - \ell$ entries equal to $s + t$ on the main diagonal, and t's elsewhere. Note that $M_{n,0} = sI + tJ$ and $M_{n,n} = tJ$, where I and J are $n \times n$. Define

$$P_{n,\ell} = \operatorname{per}(M_{n,\ell}) \tag{5.1}$$

to be the permanent of $M_{n,\ell}$.

For $\ell = 0$, define $P_{0,0} = 1$, and, recalling (2.19),

$$P_{n,0} = \operatorname{per}(sI + tJ) = \sum_{j=0}^{n} \frac{n!}{j!} s^j t^{n-j} = \sum_{j=0}^{n} \frac{n!}{(n-j)!} s^{n-j} t^j. \tag{5.2}$$

We have also $P_{n,n} = \operatorname{per}(tJ) = n!t^n$ for J of order $n \times n$. These agree at $P_{0,0} = 1$.

Theorem 5.2. *For $n \geq 1$, $1 \leq \ell \leq n$, one has the recurrence*

$$P_{n,\ell} = P_{n,\ell-1} - sP_{n-1,\ell-1}. \tag{5.3}$$

Proof. We have $0 \leq \ell \leq n$ so $n - (\ell - 1) = n - \ell + 1 \geq 1$, that is, the matrix $M_{n,\ell-1}$ contains at least 1 entry on its main diagonal equal to $s + t$. Write the block form

$$M_{n,\ell-1} = \begin{bmatrix} s+t & A \\ A^T & M_{n-1,\ell-1} \end{bmatrix}, \tag{5.4}$$

with $A = [t, t, \ldots, t]$ the $1 \times (n-1)$ row vector of all ts, and A^T is its transpose. Now, compute the permanent of $M_{n,\ell-1}$ expanding along the first row. We get

$$P_{n,\ell-1} = \operatorname{per}(M_{n,\ell-1}) = (s+t)\operatorname{per}(M_{n-1,\ell-1}) + F\Big(A, A^T, M_{n-1,\ell-1}\Big), \tag{5.5}$$

where $F(A, A^T, M_{n-1,\ell-1})$ is the contribution to $P_{n,\ell-1}$ involving A. Now,

$$\begin{aligned}
t \text{ per } (M_{n-1,\ell-1}) + F\left(A, A^T, M_{n-1,\ell-1}\right) &= \text{per}\left(\begin{bmatrix} t & A \\ A^T & M_{n-1,\ell-1} \end{bmatrix}\right) \\
&= \text{per}\left(\begin{bmatrix} A^T & M_{n-1,\ell-1} \\ t & A \end{bmatrix}\right) \\
&= \text{per}\left(\begin{bmatrix} M_{n-1,\ell-1} & A^T \\ A & t \end{bmatrix}\right) \\
&= P_{n,\ell}.
\end{aligned} \tag{5.6}$$

Thus, from (5.5),

$$\begin{aligned}
P_{n,\ell-1} &= s \text{ per } (M_{n-1,\ell-1}) + t \,\text{per}(M_{n-1,\ell-1}) + F\left(A, A^T, M_{n-1,\ell-1}\right) \\
&= sP_{n-1,\ell-1} + P_{n,\ell},
\end{aligned} \tag{5.7}$$

and hence the result. □

We arrange the polynomials $P_{n,\ell}$ in a triangle, with the columns labelled by $\ell \geq 0$ and rows by $n \geq 0$, starting with $P_{0,0} = 1$ at the top vertex

$$\begin{array}{cccccc}
P_{0,0} & & & & & \\
P_{1,0} & P_{1,1} & & & & \\
P_{2,0} & P_{2,1} & P_{2,2} & & & \\
\vdots & & \ddots & \ddots & & \\
P_{n-1,0} & \cdots & & P_{n-1,n-2} & P_{n-1,n-1} & \\
P_{n,0} & P_{n,1} & \cdots & P_{n,n-1} & P_{n,n} &
\end{array} \tag{5.8}$$

The recurrence says that to get the n, ℓ entry, you combine elements in column $\ell - 1$ in rows n and $n - 1$, forming an L-shape. Thus, given the first column $\{P_{n,0}\}_{n\geq 0}$, the table can be generated in full.

Now, we check that these are indeed our exponential moment polynomials. Additionally, we derive an expression for $P_{n,\ell}$ in terms of the initial sequence $P_{n,0}$. For clarity, we will explicitly denote the dependence of $P_{n,\ell}$ on (s, t).

Theorem 5.3. *For $\ell \geq 0$, one has*

(1) *the permanent of the $n \times n$ matrix with $n - \ell$ entries on the diagonal equal to $s + t$ and all other entries equal to t is*

$$P_{n,\ell}(s, t) = h_{n-\ell,\ell}(s, t) = \sum_{j=\ell}^{n} \binom{n-\ell}{n-j} j! s^{n-j} t^j, \tag{5.9}$$

(2)

$$P_{n,\ell}(s,t) = \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j s^j P_{n-j,0}(s,t), \tag{5.10}$$

(3) *the complementary sum is*

$$s^n = \sum_{j=0}^{n} \binom{n}{\ell} (-1)^{\ell} P_{n,\ell}(s,t). \tag{5.11}$$

Proof. The initial sequence $P_{n,0} = h_{n,0}$ as noted in (5.2). We check that $h_{n-\ell,\ell}$ satisfies recurrence (5.3). Starting from the integral representation for $h_{n-\ell+1,\ell-1}$, (3.2), we have

$$\begin{aligned} h_{n-\ell+1,\ell-1} &= \int_0^{\infty} (s+ty)^{n-\ell+1} (ty)^{\ell-1} e^{-y} dy \\ &= \int_0^{\infty} (s+ty)(s+ty)^{n-\ell} (ty)^{\ell-1} e^{-y} dy \\ &= s\ h_{n-\ell,\ell-1} + h_{n-\ell,\ell}, \end{aligned} \tag{5.12}$$

as required, where we now identify $h_{n-\ell+1,\ell-1} = P_{n,\ell-1}$, $h_{n-\ell,\ell-1} = P_{n-1,\ell-1}$, and $h_{n-\ell,\ell} = P_{n,\ell}$. And (3.10) gives an explicit form for $P_{n,\ell}$.

For (2), starting with the integral representation for $P_{n,0} = h_{n,0}$, we get

$$\begin{aligned} \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j s^j \int_0^{\infty} (s+ty)^{n-j} e^{-y} dy &= \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j s^j \int_0^{\infty} (s+ty)^{n-\ell} (s+ty)^{\ell-j} e^{-y} dy \\ &= \int_0^{\infty} (s+ty)^{n-\ell} (s+ty-s)^{\ell} e^{-y} dy \\ &= h_{n-\ell,\ell}, \end{aligned} \tag{5.13}$$

as required. The proof for (3) is similar, using (3.12),

$$P_{n,\ell} = h_{n-\ell,\ell} = \int_0^{\infty} (s+ty)^{n-\ell} (ty)^{\ell} e^{-y} dy, \tag{5.14}$$

and the binomial theorem for the sum. □

5.1. $(sI + tJ)^{\vee \ell}$ Revisited

Now, we have an alternative proof of Theorem 4.1.

Lemma 5.4. *Let* I *and* J *be* ℓ*-subsets of* $[n]$ *with* $\operatorname{dist}_{\mathrm{JS}}(\mathrm{I},\mathrm{J}) = k$*, then*

$$\operatorname{per}(sI + tJ)_{\mathrm{IJ}} = P_{\ell,k}(s,t). \tag{5.15}$$

Proof. Now, $|\mathrm{I} \cap \mathrm{J}| = \ell - k$, so the submatrix $(sI + tJ)_{\mathrm{IJ}}$ is permutationally equivalent to the $\ell \times \ell$ matrix with $\ell - k$ entries $s + t$ on its main diagonal and ts elsewhere, that is, to the matrix $M_{\ell,k}$. Hence, by definition of $P_{\ell,k}(s,t)$, (5.1), we have the result. □

Thus, the expansion in the Johnson basis is

$$(sI + tJ)^{\vee \ell} = \sum_k h_{\ell-k,k}(s,t)\mathrm{JS}_k^{n\ell}. \tag{5.16}$$

Proof. Let I and J be ℓ-subsets of $[n]$ with Johnson distance k. By definition, the IJ entry of the LHS of (5.16) equals the permanent of the submatrix from rows I and columns J, $\operatorname{per}(sI + tJ)_{\mathrm{IJ}} = P_{\ell,k}(s,t) = h_{\ell-k,k}(s,t)$, by Lemma 5.4 and Theorem 5.3(1). Now, on the RHS of (5.16), if $\operatorname{dist}_{\mathrm{JS}}(\mathrm{I},\mathrm{J}) = k$, the only nonzero contribution comes from the $\mathrm{JS}_k^{n\ell}$ term. This yields $h_{\ell-k,k}(s,t) \times 1 = h_{\ell-k,k}(s,t)$ as required. □

5.2. Elementary Subgraphs and Permanents

There is an approach to permanents of $sI + tJ$ via elementary subgraphs, based on that of Biggs [3] for determinants.

An *elementary subgraph* (see [3, page 44]) of a graph G is a spanning subgraph of G all of whose components are 0, 1, or 2 regular, that is, all of whose components are isolated vertices, isolated edges, or cycles of length $j \geq 3$.

Let $K_n^{(\ell)}$ be a copy of the complete graph K_n with vertex set $[n]$ in which the first $n - \ell$ vertices $[n - \ell] = \{1, 2, \dots, n - \ell\}$ are *distinguished*. We may now consider the matrix $M_{n,\ell}$ as the *weighted* adjacency matrix of $K_n^{(\ell)}$ in which the weights of the distinguished vertices are $s + t$, with all undistinguished vertices and all edges assigned a weight of t.

Let E be an elementary subgraph of $K_n^{(\ell)}$, then we describe E as having $d(E)$ distinguished isolated vertices and $c(E)$ cycles. The weight of E, $\operatorname{wt}(E)$, is defined as

$$\operatorname{wt}(E) = (s + t)^{d(E)} t^{n-d(E)}, \tag{5.17}$$

a homogeneous polynomial of degree n.

This leads to an interpretation/derivation of $P_{n,\ell}(s,t)$ as the permanent per $(M_{n,\ell})$.

Theorem 5.5. *One has the expansion in elementary subgraphs*

$$P_{n,\ell}(s,t) = \sum_E 2^{c(E)} \operatorname{wt}(E). \tag{5.18}$$

Proof. Assign weights to the components of E as follows:

each distinguished isolated vertex will have weight $s + t$;

each undistinguished isolated vertex will have weight t;

each isolated edge will have weight t^2;

and each j-cycle, $j \geq 3$, will have weight t^j.

To obtain $\mathrm{wt}(E)$ in agreement with (5.17), we form the product of these weights over all components in E. The proof then follows along the lines of Proposition 7.2 of [3, page 44], slightly modified to incorporate isolated vertices and with determinant, "det," replaced by permanent, "per," ignoring the minus signs. Effectively, each term in the permanent expansion thus corresponds to a weighted elementary subgraph E of the weighted $K_n^{(\ell)}$. □

See Figure 2 for an example with $n = 3$.

5.3. Associated Polynomials and Some Asymptotics

Thinking of s and t as parameters, we define the *associated polynomials*

$$Q_n(x) = \sum_{\ell=0}^{n} \binom{n}{\ell} x^{\ell} P_{n,\ell}. \tag{5.19}$$

As in the proof of (3) above, using the integral formula (3.12), we have

$$\begin{aligned} Q_n(x) &= \int_0^{\infty} (s + ty + xty)^n \, e^{-y} dy \\ &= \sum_j \binom{n}{j} s^j (1+x)^{n-j} t^{n-j} (n-j)! \\ &= n! \sum_j \frac{s^j (1+x)^{n-j} t^{n-j}}{j!}. \end{aligned} \tag{5.20}$$

Comparing with (5.2), we have the following.

Proposition 5.6. *Consider*

$$Q_n(x) = \sum_{\ell=0}^{n} \binom{n}{\ell} x^{\ell} P_{n,\ell}(s,t) = P_{n,0}(s, t + xt). \tag{5.21}$$

And one has the following.

Proposition 5.7. *As $n \to \infty$, for $x \neq -1$,*

$$Q_n(x) \sim t^n (1+x)^n n! e^{s/(t+tx)}, \tag{5.22}$$

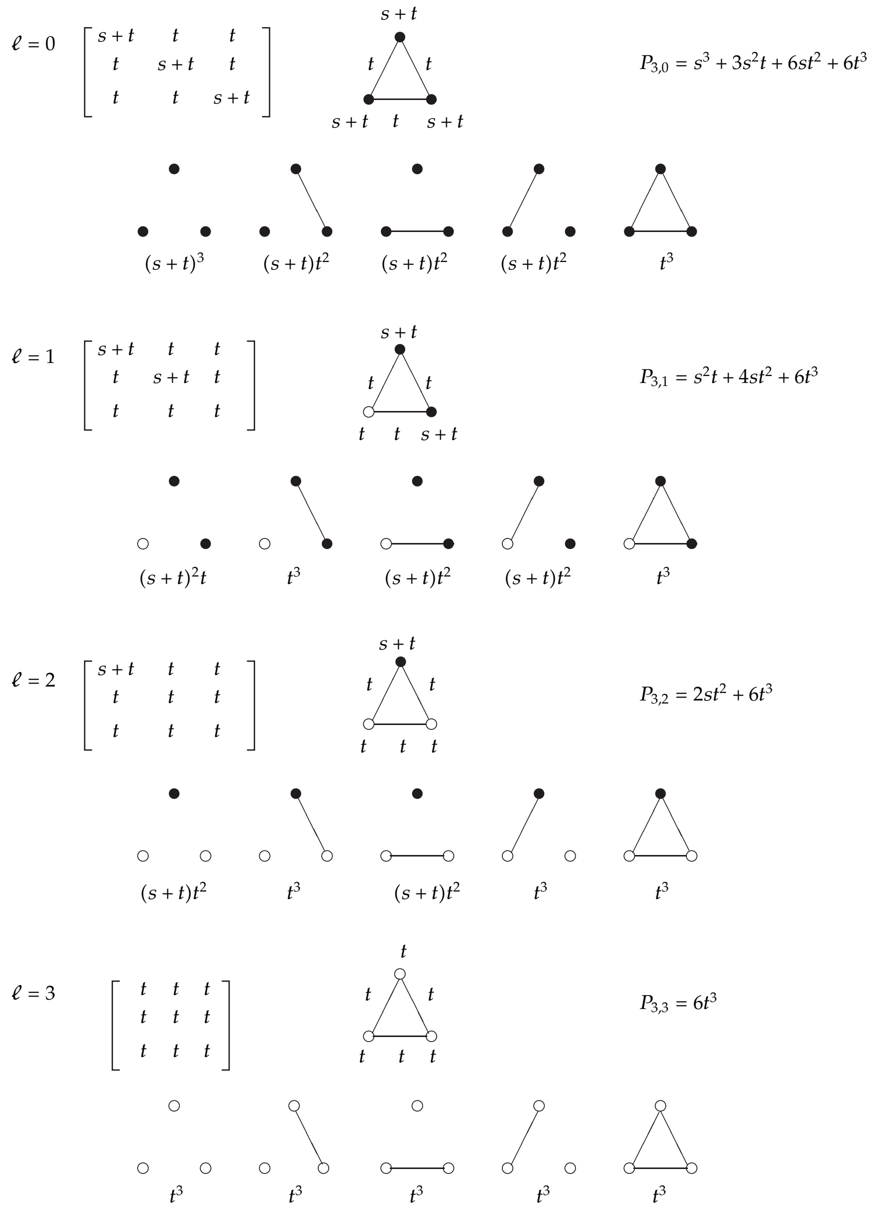

Figure 2: $M_{3,\ell}$, $K_3^{(\ell)}$, $P_{3,\ell}$, and the 5 weighted elementary subgraphs of $K_3^{(\ell)}$ for $\ell = 0,\ 1,\ 2,\ 3$. Distinguished vertices are shown in **bold**.

with the special cases

$$
\begin{aligned}
Q_n(-1) &= s^n, \\
Q_n(0) &= P_{n,0} \sim t^n n! e^{s/t}, \\
Q_n(1) &= \sum_{\ell} \binom{n}{\ell} P_{n,\ell} \sim (2t)^n n! e^{s/(2t)}.
\end{aligned}
\tag{5.23}
$$

Proof. From (5.20),

$$\begin{aligned} Q_n(x) &= n!\sum_j \frac{s^j(1+x)^{n-j}t^{n-j}}{j!} \\ &= t^n(1+x)^n n!\sum_{j=0}^{n}\frac{1}{j!}\left(\frac{s/t}{1+x}\right)^j, \end{aligned} \tag{5.24}$$

from which the result follows. □

6. Generalized Derangement Numbers

The formula (2.19) is suggestive of the derangement numbers (see, e.g., [4, page 180]),

$$d_n = n!\sum_{j=0}^{n}\frac{(-1)^j}{j!}. \tag{6.1}$$

This leads to the following.

Definition 6.1. A family of numbers, depending on n and ℓ, arising as the values of $P_{n,\ell}(s,t)$ when s and t are assigned fixed integer values, are called *generalized derangement numbers*.

We have seen that the assignment $s = -1$, $t = 1$ produces the usual derangement numbers when $\ell = 0$. In this section, we will examine in detail the cases $s = -1$, $t = 1$, generalized *derangements*, and $s = t = 1$, generalized *arrangements*.

Remark 6.2. Topics related to this material are discussed in Riordan [5]. The paper [6] is of related interest as well.

6.1. Generalized Derangements of $[n]$

To start, define

$$D_{n,\ell} = P_{n,\ell}(-1,1). \tag{6.2}$$

Equation (5.9) and Proposition 3.2 give

$$D_{n,\ell} = \sum_{j=\ell}^{n}(-1)^{n-j}\binom{n-\ell}{n-j}j! = (-1)^{n-\ell}\ \ell!\ {}_2F_0\left(\left.\begin{matrix}\ell-n, 1+\ell\\ —\end{matrix}\ \right|1\right). \tag{6.3}$$

Equation (5.2) reads as

$$\operatorname{per}(J-I) = D_{n,0} = d_n, \tag{6.4}$$

the number derangements of $[n]$. So we have a combinatorial interpretation of $D_{n,0}$.

6.1.1. Combinatorial Interpretation of $D_{n,\ell}$

We now give a combinatorial interpretation of $D_{n,\ell}$ for $\ell \geq 1$.

When $\ell \geq 1$, recurrence (5.3) for $P_{n,\ell}(-1,1)$ gives

$$D_{n,\ell} = D_{n,\ell-1} + D_{n-1,\ell-1}. \tag{6.5}$$

We say that a subset I of $[n]$ is *deranged* by a permutation if no point of I is fixed by the permutation.

Proposition 6.3. *$D_{n,0} = d_n$, the number of derangements of $[n]$. In general, for $\ell \geq 0$, $D_{n,\ell}$ is the number of permutations of $[n]$ in which the set $\{1, 2, \ldots, n-\ell\}$ is deranged, with no restrictions on the ℓ-set $\{n-\ell+1, \ldots, n\}$.*

Proof. For $\ell \geq 0$, let $D^*_{n,\ell}$ denote the set of permutations in the statement of the proposition. Let $E_{n,\ell} = |D^*_{n,\ell}|$. We claim that $E_{n,\ell} = D_{n,\ell}$.

The case $\ell = 0$ is immediate. We show that $E_{n,\ell}$ satisfies recurrence (6.5).

Now, let $\ell > 0$. Consider a permutation in $D^*_{n,\ell}$. The point n is either (1) deranged, or (2) not deranged (i.e., fixed).

(1) If n is deranged, then the $(n-\ell+1)$-set $\{1, 2, \ldots, n-\ell, n\}$ is deranged. By switching $n \leftrightarrow n-\ell+1$ in all permutations of $D^*_{n,\ell}$, we obtain a permutation in $D^*_{n,\ell-1}$. Conversely, given any permutation of $D^*_{n,\ell-1}$, we switch $n \leftrightarrow n-\ell+1$ to obtain a permutation in $D^*_{n,\ell}$ where n is deranged. Hence, the number of permutations in $D^*_{n,\ell}$ with n deranged equals $E_{n,\ell-1}$.

(2) Here, n is fixed, so if we remove n from any permutation in $D^*_{n,\ell}$ we obtain a permutation in $D^*_{n-1,\ell-1}$. Conversely, given a permutation in $D^*_{n-1,\ell-1}$, we may include n as a fixed point to obtain a permutation in $D^*_{n,\ell}$ with n fixed. Hence, the number of permutations in $D^*_{n,\ell}$ with n fixed equals $E_{n-1,\ell-1}$.

Combining the above two paragraphs shows that $E_{n,\ell}$ satisfies recurrence (6.5). □

And a quick check,

$$D_{n,n} = n!, \tag{6.6}$$

there being no restrictions at all in the combinatorial interpretation, in agreement with (6.3) for $\ell = n$.

Example 6.4. When $n = 3$, we have $d_3 = D_{3,0} = 2$ corresponding to the 2 permutations of [1] in which $\{1,2,3\}$ is moved: $231, 312$.

Then, $D_{3,1} = 3$ corresponding to the 3 permutations of [1] in which $\{1,2\}$ is moved: $213, 231, 312$.

Then, $D_{3,2} = 4$ corresponding to the 4 permutations of [1] in which $\{1\}$ is moved: $213, 231, 312, 321$.

Finally, $D_{3,3} = 3! = 6$ corresponding to the 3 permutations of [1] in which $\emptyset$ is moved: $123, 132, 213, 231, 312, 321$.

Reversing the order of summation in (6.3) gives an alternative expression

$$D_{n,\ell} = \sum_{j=0}^{n-\ell} (-1)^j \binom{n-\ell}{j} (n-j)!. \tag{6.7}$$

Remark 6.5. Formulation (6.7) may be proved directly by inclusion-exclusion on permutations fixing given points.

Example 6.6. Consider

$$\begin{aligned} D_{5,2} &= \sum_{j=0}^{3} (-1)^j \binom{3}{j} (5-j)! = \binom{3}{0}5! - \binom{3}{1}4! + \binom{3}{2}3! - \binom{3}{3}2! \\ &= 120 - 72 + 18 - 2 = 64. \end{aligned} \tag{6.8}$$

Now, from (2) of Theorem 5.3, $s = -1$, and $t = 1$, we have

$$D_{n,\ell} = \sum_{j=0}^{\ell} \binom{\ell}{j} d_{n-j}. \tag{6.9}$$

Here is a combinatorial explanation. To obtain a permutation in $D^*_{n,\ell}$, we first choose j points from $\{n-\ell+1,\dots,n\}$ to be fixed. Then, every derangement of the remaining $(n-j)$ points will produce a permutation in $D^*_{n,\ell}$, and there are d_{n-j} of such derangements.

Example 6.7. Consider

$$\begin{aligned} D_{5,2} &= \sum_{j=0}^{2} \binom{2}{j} d_{5-j} = \binom{2}{0}d_5 + \binom{2}{1}d_4 + \binom{2}{2}d_3 \\ &= 1 \times 44 + 2 \times 9 + 1 \times 2 = 44 + 18 + 2 = 64. \end{aligned} \tag{6.10}$$

6.1.2. *Permanents from $J - I$*

Theorem 4.1 specializes to

$$(J-I)^{\vee\ell} = \sum_{k=0}^{\min(\ell,n-\ell)} D_{\ell,k}\, \mathrm{JS}_k^{n\ell}. \tag{6.11}$$

This can be written using the hypergeometric form

$$(J-I)^{\vee\ell} = \sum_{k=0}^{\min(\ell,n-\ell)} (-1)^{\ell-k}\, k!\, {}_2F_0\left(\left. \begin{matrix} k-\ell, 1+k \\ - \end{matrix} \right| 1 \right) \mathrm{JS}_k^{n\ell}, \tag{6.12}$$

with spectrum

$$\begin{gathered}\text{eigenvalue } (-1)^{\ell}\ {}_2F_0\left(\begin{matrix}\alpha-\ell,-\alpha+n-\ell+1\\ -\end{matrix}\,\middle|\,1\right),\\ \text{occurring with multiplicity}\binom{n}{\alpha}-\binom{n}{\alpha-1},\end{gathered} \tag{6.13}$$

by Corollary 4.3 and Proposition 4.2.

The entries of $(J-I)^{\vee\ell}$ are from the set of numbers $D_{n,\ell}$. For the spectrum, start with $\alpha=0$. From (6.3), we have

$$(-1)^{\ell}\ {}_2F_0\left(\begin{matrix}-\ell,n-\ell+1\\ -\end{matrix}\,\middle|\,1\right)=\frac{1}{(n-\ell)!}D_{n,n-\ell}. \tag{6.14}$$

As α increases, we see that the spectrum consists of the numbers

$$\frac{(-1)^{\alpha}}{(n-\ell-\alpha)!}D_{n-2\alpha,n-\ell-\alpha}. \tag{6.15}$$

Think of moving in the derangement triangle, as in the appendix, starting from position $n,n-\ell$, rescaling the values by the factorial of the column at each step, then the eigenvalues are found by successive knight's moves, up 2 rows and one column to the left, with alternating signs.

Example 6.8. For $n=5$, $\ell=3$, we have

$$(J-I)^{\vee 3}=\begin{bmatrix}2&3&3&3&3&4&3&3&4&4\\3&2&3&3&4&3&3&4&3&4\\3&3&2&4&3&3&4&3&3&4\\3&3&4&2&3&3&3&4&4&3\\3&4&3&3&2&3&4&3&4&3\\4&3&3&3&3&2&4&4&3&3\\3&3&4&3&4&4&2&3&3&3\\3&4&3&4&3&4&3&2&3&3\\4&3&3&4&4&3&3&3&2&3\\4&4&4&3&3&3&3&3&3&2\end{bmatrix}, \tag{6.16}$$

with characteristic polynomial

$$\lambda^5(\lambda-32)(\lambda+3)^4. \tag{6.17}$$

Remark 6.9. Except for $\ell = 2$, the coefficients in the expansion of $(J - I)^{\vee\ell}$ in the Johnson basis will be distinct. Thus, the Johnson basis itself can be read off directly from $(J - I)^{\vee\ell}$. In this sense, the centralizer algebra of the action of the symmetric group on ℓ-sets is determined by knowledge of the action of just $J - I$ on ℓ-sets.

6.2. Generalized Arrangements of $[n]$

Given $[n]$, $0 \le j \le n$, a *j-arrangement* of $[n]$ is a permutation of a j-subset of $[n]$. The number of j-arrangements of $[n]$ is

$$A(n,j) = \frac{n!}{(n-j)!}. \tag{6.18}$$

Note that there is a single 0-arrangement of $[n]$, from the empty set.

Define $A_{n,\ell} = P_{n,\ell}(1,1)$. So, similar to the case for derangements, (5.9) gives

$$A_{n,\ell} = \sum_{j=\ell}^{n} \binom{n-\ell}{n-j} j! = \ell!\ {}_2F_0\left(\begin{matrix} \ell - n, 1+\ell \\ — \end{matrix} \middle| -1 \right). \tag{6.19}$$

Now, define $a_n = A_{n,0}$, so

$$a_n = \operatorname{per}(I + J) = \sum_{j=0}^{n} \frac{n!}{(n-j)!} = \sum_{j=0}^{n} A(n,j) \tag{6.20}$$

is the *total* number of j-arrangements of $[n]$ for $j = 0, 1, \ldots, n$. Thus, we have a combinatorial interpretation of $A_{n,0}$.

6.2.1. Combinatorial Interpretation of $A_{n,\ell}$

We now give a combinatorial interpretation of $A_{n,\ell}$ for $\ell \ge 1$.

When $\ell \ge 1$, recurrence (5.3) for $P_{n,\ell}(1,1)$ gives

$$A_{n,\ell} = A_{n,\ell-1} - A_{n-1,\ell-1}. \tag{6.21}$$

Proposition 6.10. *$A_{n,0} = a_n$, the total number of arrangements of $[n]$. In general, for $\ell \ge 0$, $A_{n,\ell}$ is the number of arrangements of $[n]$ which contain $\{1, 2, \ldots, \ell\}$.*

Proof. For $\ell \ge 0$, let $A^*_{n,\ell}$ denote the set of arrangements of $[n]$ which contain $[\ell]$. With $[0] = \emptyset$, we note that $A^*_{n,0}$ is the set of all arrangements. Let $B_{n,\ell} = |A^*_{n,\ell}|$. We claim that $B_{n,\ell} = A_{n,\ell}$.

The initial values with $\ell = 0$ are immediate. We show that $B_{n,\ell}$ satisfies recurrence (6.21).

Consider $A^*_{n,\ell-1}$. Let $A \in A^*_{n,\ell-1}$, so A is an arrangement of $[n]$ containing $[\ell - 1]$. If $\ell = 1$, then $A \in A^*_{n,0}$ is any arrangement. Now, either $\ell \in A$ or $\ell \notin$ A.

If $\ell \in A$, then $A \in A^*_{n,\ell}$, and so the number of arrangements in $A^*_{n,\ell-1}$ which contain ℓ equals $B_{n,\ell}$.

If $\ell \notin$ A, then by subtracting 1 from all parts of A which are $\geq \ell + 1$, we obtain an arrangement of $[n-1]$ which contains $[\ell-1]$, that is, an arrangement in $A^*_{n-1,\ell-1}$. Conversely, given an arrangement in $A^*_{n-1,\ell-1}$, adding 1 to all parts $\geq \ell$ yields an arrangement in $A^*_{n,\ell-1}$ which does not contain ℓ. Hence, the number of arrangements in $A^*_{n,\ell-1}$ which do not contain ℓ equals $B_{n-1,\ell-1}$.

We conclude that $B_{n,\ell-1} = B_{n,\ell} + B_{n-1,\ell-1}$; hence, This is the result. □

Example 6.11. When $n = 3$, we have $a_3 = A_{3,0} = 16$ corresponding to the 16 arrangements of [1]: [ ], 1, 2, 3, 12, 21, 13, 31, 23, 32, 123, 132, 213, 231, 312, 321.

Then, $A_{3,1} = 11$ corresponding to the 11 arrangements of [1] which contain $\{1\}$: 1, 12, 21, 13, 31, 123, 132, 213, 231, 312, 321.

Then, $A_{3,2} = 8$ corresponding to the 8 arrangements of [1] which contain $\{1,2\}$: 12, 21, 123, 132, 213, 231, 312, 321.

Finally, $A_{3,3} = 3! = 6$ corresponding to the 6 arrangements of [1] which contain $\{1,2,3\}$: 123, 132, 213, 231, 312, 321.

Rearranging the factors in (5.9), we have

$$P_{n,\ell}(s,t) = \sum_{j=\ell}^{n} A(j,\ell)A(n-\ell,j-\ell)s^{n-j}t^{j}, \tag{6.22}$$

With $s = t = 1$, this gives

$$A_{n,\ell} = \sum_{j=\ell}^{n} A(j,\ell)A(n-\ell,j-\ell). \tag{6.23}$$

Here is a combinatorial explanation of (6.23).

For any $j \geq \ell$, to obtain a j-arrangement A of $[n]$ containing $[\ell]$, we may place the ℓ points of $\{1,2,\ldots,\ell\}$ into these j positions in $A(j,\ell)$ ways. Then, the remaining $(j-\ell)$ positions in A can be filled in by a $(j-\ell)$-arrangement of the unused $(n-\ell)$ points in $A(n-\ell,j-\ell)$ ways.

Example 6.12. Consider

$$\begin{aligned}
A_{5,2} &= \sum_{j=2}^{5} A(j,2)\ A(3,j-2) \\
&= A(2,2)\ A(3,0) + A(3,2)\ A(3,1) + A(4,2)\ A(3,2) + A(5,2)\ A(3,3) \\
&= 2\times 1 + 6\times 3 + 12\times 6 + 20\times 6 \\
&= 2 + 18 + 72 + 120 = 212.
\end{aligned} \tag{6.24}$$

Finally, from (2) of Theorem 5.3, $s = 1$, and $t = 1$, we have

$$A_{n,\ell} = \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} a_{n-j}. \tag{6.25}$$

Example 6.13. Consider

$$\begin{aligned} A_{5,2} &= \sum_{j=0}^{2} (-1)^j \binom{2}{j} a_{5-j} = \binom{2}{0} a_5 - \binom{2}{1} a_4 + \binom{2}{2} a_3 \\ &= 1 \times 326 - 2 \times 65 + 1 \times 16 = 326 - 130 + 16 = 212. \end{aligned} \tag{6.26}$$

6.2.2. *Permanents from $I + J$*

Theorem 4.1 specializes to

$$(I + J)^{\vee \ell} = \sum_{k=0}^{\min(\ell, n-\ell)} A_{\ell,k} \mathrm{JS}_k^{n\ell}. \tag{6.27}$$

This can be written using the hypergeometric form

$$(I + J)^{\vee \ell} = \sum_{k=0}^{\min(\ell, n-\ell)} k!\, {}_2F_0\left(\left. \begin{matrix} k - \ell, 1 + k \\ — \end{matrix} \right| -1 \right) \mathrm{JS}_k^{n\ell}, \tag{6.28}$$

with spectrum

$$\begin{aligned} &\text{eigenvalue } {}_2F_0\left(\left. \begin{matrix} \alpha - \ell, -\alpha + n - \ell + 1 \\ — \end{matrix} \right| -1 \right), \\ &\text{occurring with multiplicity} \binom{n}{\alpha} - \binom{n}{\alpha - 1}, \end{aligned} \tag{6.29}$$

by Corollary 4.3 and Proposition 4.2.

Example 6.14. For $n = 5$, $\ell = 3$, we have

$$(I+J)^{\vee 3} = \begin{bmatrix} 16 & 11 & 11 & 11 & 11 & 8 & 11 & 11 & 8 & 8 \\ 11 & 16 & 11 & 11 & 8 & 11 & 11 & 8 & 11 & 8 \\ 11 & 11 & 16 & 8 & 11 & 11 & 8 & 11 & 11 & 8 \\ 11 & 11 & 8 & 16 & 11 & 11 & 11 & 8 & 8 & 11 \\ 11 & 8 & 11 & 11 & 16 & 11 & 8 & 11 & 8 & 11 \\ 8 & 11 & 11 & 11 & 11 & 16 & 8 & 8 & 11 & 11 \\ 11 & 11 & 8 & 11 & 8 & 8 & 16 & 11 & 11 & 11 \\ 11 & 8 & 11 & 8 & 11 & 8 & 11 & 16 & 11 & 11 \\ 8 & 11 & 11 & 8 & 8 & 11 & 11 & 11 & 16 & 11 \\ 8 & 8 & 8 & 11 & 11 & 11 & 11 & 11 & 11 & 16 \end{bmatrix}, \tag{6.30}$$

with characteristic polynomial

$$(\lambda - 106)(\lambda - 11)^4(\lambda - 2)^5. \tag{6.31}$$

As for the case of derangements, the Johnson basis can be read off directly from the matrix $(I+J)^{\vee \ell}$.

Appendix

Generalized Derangement Numbers and Integer Sequences

The first two columns of the $D_{n,\ell}$ triangle, $D_{n,0}$ and $D_{n,1}$, give sequences A000166 and A000255 in the On-Line Encyclopedia of Integer Sequences [7]. The comments for A000255 do not contain our combinatorial interpretation.

The first two columns of the $A_{n,\ell}$ triangle, $A_{n,0}$ and $A_{n,1}$, give sequences A000522 and A001339. The comments contain our combinatorial interpretation. The next two columns, $A_{n,2}$ and $A_{n,3}$, give sequences A001340 and A00134; here, our combinatorial interpretation is not mentioned in the comments.

Generalized Derangement Triangles

$\ell = 0$ is the leftmost column. The rows correspond to n from 0 to 9.

Values of $D_{n,\ell}$

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 3 & 4 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
9 & 11 & 14 & 18 & 24 & 0 & 0 & 0 & 0 & 0 \\
44 & 53 & 64 & 78 & 96 & 120 & 0 & 0 & 0 & 0 \\
265 & 309 & 362 & 426 & 504 & 600 & 720 & 0 & 0 & 0 \\
1854 & 2119 & 2428 & 2790 & 3216 & 3720 & 4320 & 5040 & 0 & 0 \\
14833 & 16687 & 18806 & 21234 & 24024 & 27240 & 30960 & 35280 & 40320 & 0 \\
133496 & 148329 & 165016 & 183822 & 205056 & 229080 & 256320 & 287280 & 322560 & 362880
\end{bmatrix}. \tag{A.1}$$

Values of $A_{n,\ell}$

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
16 & 11 & 8 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
65 & 49 & 38 & 30 & 24 & 0 & 0 & 0 & 0 & 0 \\
326 & 261 & 212 & 174 & 144 & 120 & 0 & 0 & 0 & 0 \\
1957 & 1631 & 1370 & 1158 & 984 & 840 & 720 & 0 & 0 & 0 \\
13700 & 11743 & 10112 & 8742 & 7584 & 6600 & 5760 & 5040 & 0 & 0 \\
109601 & 95901 & 84158 & 74046 & 65304 & 57720 & 51120 & 45360 & 40320 & 0 \\
986410 & 876809 & 780908 & 696750 & 622704 & 557400 & 499680 & 448560 & 403200 & 362880
\end{bmatrix}. \tag{A.2}$$

Exponential polynomials $h_{n,m}(s,t)$

Note that, as is common for matrix indexing, we have dropped the commas in the numerical subscripts

$n = 0$

$$h_{00} = 1, \quad h_{01} = t, \quad h_{02} = 2t^2, \quad h_{03} = 6t^3, \quad h_{04} = 24t^4, \tag{A.3}$$

$n = 1$

$$h_{10} = s + t, \quad h_{11} = st + 2t^2, \quad h_{12} = 2st^2 + 6t^3, \quad h_{13} = 6st^3 + 24t^4, \quad h_{14} = 24st^4 + 120t^5, \tag{A.4}$$

$n = 2$

$$\begin{gathered} h_{20} = s^2 + 2st + 2t^2, \quad h_{21} = s^2t + 4st^2 + 6t^3, \quad h_{22} = 2s^2t^2 + 12st^3 + 24t^4, \\ h_{23} = 6s^2t^3 + 48st^4 + 120t^5, \quad h_{24} = 24s^2t^4 + 240st^5 + 720t^6, \end{gathered} \tag{A.5}$$

$n = 3$

$$\begin{gathered} h_{30} = s^3 + 3s^2t + 6st^2 + 6t^3, \\ h_{31} = s^3t + 6s^2t^2 + 18st^3 + 24t^4, \quad h_{32} = 2s^3t^2 + 18s^2t^3 + 72st^4 + 120t^5, \\ h_{33} = 6s^3t^3 + 72s^2t^4 + 360st^5 + 720t^6, \quad h_{34} = 24s^3t^4 + 360s^2t^5 + 2160st^6 + 5040t^7, \end{gathered} \tag{A.6}$$

$n = 4$

$$\begin{gathered} h_{40} = s^4 + 4s^3t + 12s^2t^2 + 24st^3 + 24t^4, \quad h_{41} = s^4t + 8s^3t^2 + 36s^2t^3 + 96st^4 + 120t^5, \\ h_{42} = 2s^4t^2 + 24s^3t^3 + 144s^2t^4 + 480st^5 + 720t^6, \\ h_{43} = 6s^4t^3 + 96s^3t^4 + 720s^2t^5 + 2880st^6 + 5040t^7, \\ h_{44} = 24s^4t^4 + 480s^3t^5 + 4320s^2t^6 + 20160st^7 + 40320t^8. \end{gathered} \tag{A.7}$$

Acknowledgment

The authors would like to thank Stacey Staples for discussions about zeons and trace formulas.

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