Smallest labelled class and largest automorphism group of a tree $\widetilde{T}_{s,t}$

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Abstract

We show that the smallest class of labelled copies of an unlabelled tree $\widetilde{T}_{s,t}$ with $2 \leq s \leq t$, $(\{s,t\} \neq \{2,3\})$, in the complete bipartite graph $K_{s,t}$ has size st, is unique and has representative unlabelled tree the double-star, $D_{s,t}$. Equivalently, the tree $\widetilde{T}_{s,t}$ which has largest automorphism group size amongst all such trees is $D_{s,t}$ with automorphism group size $(s-1)!(t-1)!$ and automorphism group $\mathrm{Sym}(s-1) \times \mathrm{Sym}(t-1)$. Slight modifications of these statements are needed if $s = t$. We also produce a novel method for finding all labelled copies of tree $\widetilde{T}_{s,t}$.

Keywords: trees, $T_{s,t}$, smallest labelled class, largest automorphism group

1 Introduction and main results

In this paper G_n will denote a labelled simple graph with $n \geq 3$ vertices, and with vertex-set $V(G_n)$ and edge-set $E(G_n)$, and $deg(v)$ will denote the degree of vertex $v \in V(G_n)$. We use Cameron [1] and West [10] as general references.

Let $K_{s,t}$, where $1 \leq s \leq t$, denote the complete bipartite graph with s vertices in the first part and t vertices in the second.

Let $\widetilde{T}_n$ denote an *unlabelled* tree with n vertices, *i.e.*, an unlabelled spanning tree of the complete graph K_n. Then by Theorem 1.1 of Porter [7], there is a unique partition of $V(\widetilde{T}_n)$ into 2 parts, the first X of size s and the second Y of size t, where $1 \leq s \leq t$ and $n = s+t$. If $s = t$ we arbitrarily choose one of the parts to be X, and let the other be Y. Thus each $\widetilde{T}_n$ can be considered as a sub-tree of $K_{s,t}$ for some unique s and t with $1 \leq s \leq t$ and $n = s + t$. We then denote $\widetilde{T}_n$ by $\widetilde{T}_{s,t}$; and we denote the number of non-isomorphic $\widetilde{T}_{s,t}$ by $\widetilde{\tau}(\widetilde{T}_{s,t})$, or by $\widetilde{\tau}(K_{s,t})$ because this counts the number of non-isomorphic unlabelled spanning trees in $K_{s,t}$.

Example 1 See p.65 of Read and Wilson [8]. For $n = 7$, there are 11 non-isomorphic or distinct unlabelled trees $\widetilde{T}_7$, *i.e.*, there are 11 unlabelled spanning trees of K_7: 1 is a sub-tree of $K_{1,6}$, so $\widetilde{\tau}(\widetilde{T}_{1,6}) = 1$; 3 are sub-trees of $K_{2,5}$, so $\widetilde{\tau}(\widetilde{T}_{2,5}) = 3$; and 7 are sub-trees of $K_{3,4}$, so $\widetilde{\tau}(\widetilde{T}_{3,4}) = 7$.

An *automorphism* of G_n is a bijection $\alpha : V(G_n) \to V(G_n)$ for which $(u, v) \in E(G_n)$ if and only if $\alpha(u, v) = (\alpha(u), \alpha(v)) \in E(G_n)$, for vertices $u, v \in V(G_n)$. The set of automorphisms of G_n is a group, denoted by $\mathrm{Aut}(G_n)$; let $a(G_n) = |\,\mathrm{Aut}(G_n)|$ be the size of this group.

Let $\mathrm{Sym}(n)$ be the symmetric group on n points, and let $\times$ denote the direct product of groups, and $\boxtimes$ denote the wreath product of permutation groups.

We use the following facts from Proposition 2.1 on pp.91/2 of Cameron [2] (see Cameron [3] for an updated version of this paper), and Example 1.1.21 on p.10 of [10]: $\mathrm{Aut}(K_n) = \mathrm{Sym}(n)$ so $a(K_n) = n!$; and if $s \neq t$ then $\mathrm{Aut}(K_{s,t}) = \mathrm{Sym}(s) \times \mathrm{Sym}(t)$ so $a(K_{s,t}) = s!t!$; and $\mathrm{Aut}(K_{s,s}) = \mathrm{Sym}(s) \boxtimes \mathrm{Sym}(2)$ so $a(K_{s,s}) = 2s!^2$.

We denote *labelled* copies of $\widetilde{T}_{s,t}$ by $T_{s,t}$. It is well-known that the number of non-isomorphic or distinct labelled spanning trees $T_{s,t}$ in $K_{s,t}$ is $\tau(K_{s,t}) = s^{t-1}t^{s-1}$, see Scoins [9].

The labelled isomorphism classes for a fixed unlabelled $\widetilde{T}_{s,t}$ in $K_{s,t}$ have been generated in Porter [6], and enumerated and drawn for $s + t \leq 12$ in Mohr [5]. See also Clark, Mohr, and Porter [4].

For the derivation of Equation (1) and the counting in Example 2 below see Theorem 14.3.4 on p.234 of [1], where we replace $n!$ by $s!t!$ if $s \neq t$, and by $2s!^2$ if $s = t$.

Let $\ell(\widetilde{T}_{s,t})$ denote the number of labelled copies of an unlabelled $\widetilde{T}_{s,t}$ in $K_{s,t}$, this is the size of the labelled isomorphism class represented by $\widetilde{T}_{s,t}$. We have:

$$\ell(\widetilde{T}_{s,t}) = \frac{a(K_{s,t})}{a(\widetilde{T}_{s,t})} = \begin{cases} \dfrac{s!t!}{a(\widetilde{T}_{s,t})} & s \neq t, \\[2ex] \dfrac{2s!^2}{a(\widetilde{T}_{s,t})} & s = t. \end{cases} \tag{1}$$

For a fixed $n \geq 3$, if $s = 1$ there is only one unlabelled $\widetilde{T}_{1,n-1}$ up to isomorphism, namely the star $K_{1,n-1}$, and so only one labelled class, of size 1. Thus we consider $s \geq 2$.

Example 2 $\{s,t\} = \{3,4\}$. See [5, 8] and Fig. 1. The graph $K_{3,4}$ has $\widetilde{\tau}(K_{3,4}) = 7$ non-isomorphic unlabelled spanning trees: T17, T19, T21, T22, T23, T24, and T25; see p.65 of [8]. The automorphism group sizes $a(\widetilde{T}_{3,4})$ come from p. 116 of [8], and then $\ell(\widetilde{T}_{3,4})$ is obtained from Equation (1) above. We list them below in order of increasing $\ell(\widetilde{T}_{3,4})$:

$\widetilde{T}_{3,4}$	T17	T24	T19	T21	T22	T25	T23
$a(\widetilde{T}_{3,4})$	12	6	4	2	2	2	1
$\ell(\widetilde{T}_{3,4})$	**12**	24	36	72	72	72	144

(We check: $K_{3,4}$ has $\tau(K_{3,4}) = 3^{4-1}4^{3-1} = 432$ non-isomorphic labelled spanning trees, and $12 + 24 + 36 + 72 + 72 + 72 + 144 = 432$.)

We observe that the smallest class size $\ell(\widetilde{T}_{3,4})$ is $\mathbf{12} = 3 \times 4$, which occurs just once, and that $\mathbf{12} \,|\, \ell(\widetilde{T}_{3,4})$ for each $\widetilde{T}_{3,4}$.

In §2 we present the first of our main results, Theorems 2.4 and 2.5, of which the above observation is an example. Theorem 2.4 states that: for every pair $\{s,t\}$ with $2 \leq s \leq t$, ($\{s,t\} \neq \{2,3\}$), the smallest class of labelled trees in $K_{s,t}$ is unique and contains st trees, and the representative unlabelled tree is the double-star, $D_{s,t}$, see Fig. 2(a). And Theorem 2.5 states that: for every pair $\{s,t\}$ with $2 \leq s \leq t$ let $\widetilde{T}_{s,t}$ denote an unlabelled tree in $K_{s,t}$. Then $st \,|\, \ell(\widetilde{T}_{s,t})$. We also show how to construct the st labelled trees in the smallest class.

Now smallest class size is equivalent to largest automorphism group size, thus, in §2.1, we show that, for $s \neq t$, ($\{s,t\} \neq \{2,3\}$), the tree $\widetilde{T}_{s,t}$ which has largest automorphism group size amongst all such trees is $D_{s,t}$ with automorphism group size $(s-1)!(t-1)!$ and automorphism group $\mathrm{Sym}(s-1) \times \mathrm{Sym}(t-1)$; and if $s = t$, the tree $\widetilde{T}_{s,s}$ which has largest automorphism group size amongst all such trees is $D_{s,s}$ with automorphism group size $2(s-1)!^2$ and automorphism group $\mathrm{Sym}(s-1) \boxtimes \mathrm{Sym}(2)$, see Theorems 2.6 and 2.7.

We finish with some miscellaneous remarks and indicate directions for possible future research.

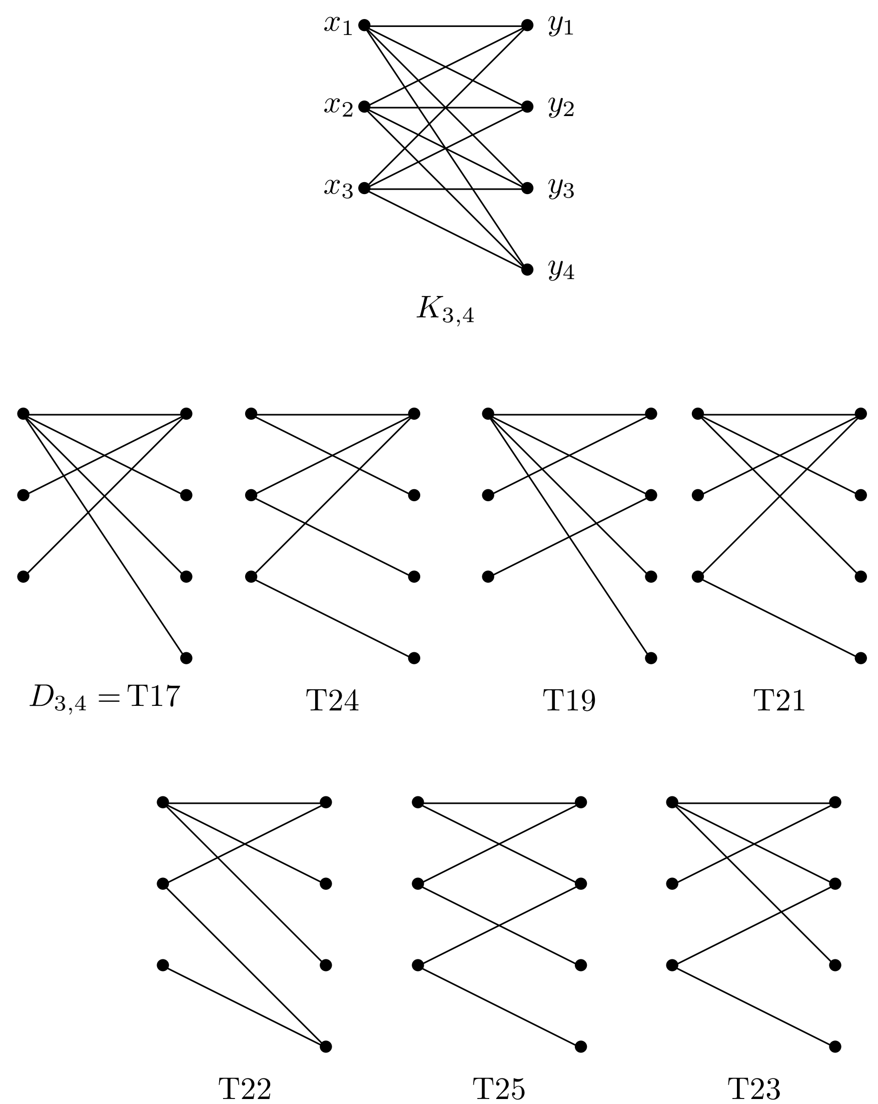


Figure 1: $K_{3,4}$ and its 7 non-isomorphic unlabelled trees $T_{3,4}$, shown with increasing $\ell(\widetilde{T}_{3,4})$.

2 Smallest labelled class in $K_{s,t}$; largest automorphism group amongst $\widetilde{T}_{s,t}$

In this section we work with a copy of $K_{s,t}$ with partite sets X and Y. Let vertices $X = \mathbb{Z}_s = \{0_X, 1, \ldots, s-1\}$ and vertices $Y = \mathbb{Z}_t = \{0_Y, 1, \ldots, t-1\}$; and $E(K_{s,t}) = \mathbb{Z}_s \times \mathbb{Z}_t$. Let $T_{s,t}$ be a labelled copy of $\widetilde{T}_{s,t}$ in $K_{s,t}$; an edge in $T_{s,t}$ is denoted $(i,j) \in \mathbb{Z}_s \times \mathbb{Z}_t$. We often use $=$ in place of $\equiv$ when using modular arithmetic.

Definition For an ordered pair $(a,b) \in \mathbb{Z}_s \times \mathbb{Z}_t$ define:

$$E(T_{s,t}) + (a,b) = \{(i + a \pmod s), j + b \pmod t)) \,|\, (i,j) \in E(T_{s,t})\}.$$

Further, let $o(i)$ denote the (additive) order of element i in the group $(\mathbb{Z}_s, +)$, and let $\langle i \rangle$ denote the subgroup of $\mathbb{Z}_s$ generated by i; similarly for element $j \in \mathbb{Z}_t$ and element $(i,j) \in \mathbb{Z}_s \times \mathbb{Z}_t$.

Recall that two labelled trees are distinct when their edge-sets are distinct.

Lemma 2.1 *For every pair $\{s,t\}$ with $2 \le s \le t$ let $\widetilde{T}_{s,t}$ be an unlabelled spanning tree of $K_{s,t}$. Then $\ell(\widetilde{T}_{s,t}) \ge st$.*

Proof. Consider a copy of $K_{s,t}$ labelled as above. Without loss of generality let $(0_X, 0_Y) \in E(T_{s,t})$. We claim that the st labelled trees with edge-set $E(T_{s,t}) + (a,b)$ for distinct $(a,b) \in \mathbb{Z}_s \times \mathbb{Z}_t$ are distinct labelled trees. That is, we claim $E(T_{s,t}) + (a,b) \ne E(T_{s,t}) + (a',b')$ when $(a,b) \ne (a',b')$.

Suppose to the contrary that $E(T_{s,t}) + (a,b) = E(T_{s,t}) + (a',b')$ for some $(a,b) \ne (a',b')$, and then $E(T_{s,t}) + (a-a', b-b') = E(T_{s,t})$. Let $(a-a', b-b') = (i_1, j_1) \ne (0_X, 0_Y)$, for a fixed $(i_1, j_1) \in \mathbb{Z}_s \times \mathbb{Z}_t$, so $E(T_{s,t}) + (i_1, j_1) = E(T_{s,t})$. Now $(0_X, 0_Y) \in E(T_{s,t})$ so $(0_X, 0_Y) + (i_1, j_1) = (i_1, j_1) \in E(T_{s,t})$, *i.e.*, (i_1, j_1) is an edge of $E(T_{s,t})$.

By iterating the equation $E(T_{s,t}) + (i_1, j_1) = E(T_{s,t})$, we see that $E(T_{s,t}) + (i,j) = E(T_{s,t})$ for all $(i,j) \in \langle (i_1, j_1) \rangle$, and that $\langle (i_1, j_1) \rangle \subseteq E(T_{s,t})$.

Consider the two cases (i) $o(i_1) \ne o(j_1)$, and (ii) $o(i_1) = o(j_1)$. In both cases we find a contradiction.

(i) $o(i_1) \ne o(j_1)$. Let $o(i_1) < o(j_1)$. Then, in $\mathbb{Z}_s \times \mathbb{Z}_t$ we have, $o(i_1)(i_1, j_1) = (o(i_1)i_1, o(i_1)j_1) = (0_X, o(i_1)j_1) = (0_X, j_2)$ where $j_2 \ne 0_Y$. And from above we know that $E(T_{s,t}) + (0_X, j_2) = E(T_{s,t})$.

Let $N(0_X) = \{j \in Y \,|\, (0_X, j) \in E(T_{s,t})\}$ be the open neighborhood of 0_X.

If $deg(j) = 1$ for every vertex $j \in N(0_X)$ then $T_{s,t}$ is the star $K_{1,|N(0_X)|}$, a contradiction because $s \geq 2$. So, for some $j \in N(0_X)$, say j_3, and some $i_2 \in \mathbb{Z}_s \backslash \{0_X\}$ the pair $(i_2, j_3) \in E(T_{s,t})$. Then the pair $(i_2, j_3) + (0_X, j_2) = (i_2, j_3 + j_2) \in E(T_{s,t})$ also. But $(0_X, j_3) \in E(T_{s,t})$ and so $(0_X, j_3) + (0_X, j_2) = (0_X, j_3 + j_2) \in E(T_{s,t})$ also.

So both vertices $0_X, i_2 \in X$ are adjacent to both vertices $j_3, j_3 + j_2 \in Y$, and then $(0_X, j_3, i_2, j_3 + j_2, 0_X)$ is a 4-cycle in $T_{s,t}$, a contradiction because $T_{s,t}$ is acylic.

The sub-case $o(i_1) > o(j_1)$ is dealt with similarly.

(ii) $o(i_1) = o(j_1)$.

Define a relation $\sim$ on $E(T_{s,t})$:

$$(i, j) \sim (i', j') \quad \text{if and only if} \quad (i - i', j - j') \in \langle (i_1, j_1) \rangle.$$

Using the fact that $\langle (i_1, j_1) \rangle$ is a group under $+$ it is straightforward to show that $\sim$ is an equivalence relation on $E(T_{s,t})$. Clearly $E_1 = \langle (i_1, j_1) \rangle \subseteq E(T_{s,t})$ is one equivalence class of $\sim$, the class containing $(0_X, 0_Y) \in E(T_{s,t})$. The $m \geq 1$ equivalence classes, $E_1, E_2, \ldots, E_m$, are given by $E_h = E_1 + (i^*, j^*) \subseteq E(T_{s,t})$ for suitable representatives $(i^*, j^*) \in E(T_{s,t})$; and all E_h have the same size. Now these equivalence classes partition $E(T_{s,t})$, and so $|E(T_{s,t})| = m|E_1|$, *i.e.*, $s + t - 1 = mo((i_1, j_1)) = mo(i_1) = mo(j_1)$, so $o(i_1) \mid (s + t - 1)$.

Further, using Lagrange's Theorem twice, we see that $o(i_1)|s$ and $o(i_1) = o(j_1)|t$, so $o(i_1)|1$, *i.e.*, $o(i_1) = 1$, and so $i_1 = 0_X$. Similarly $j_1 = 0_Y$, and then $(i_1, j_1) = (0_X, 0_Y)$, a contradiction.

Thus, in both cases (i) and (ii), we have found a contradiction to the existence of $(i_1, j_1) \in \mathbb{Z}_s \times \mathbb{Z}_t$ for which $E(T_{s,t}) + (i_1, j_1) = E(T_{s,t})$. So $E(T_{s,t}) + (a, b) \neq E(T_{s,t}) + (a', b')$ when $(a, b) \neq (a', b')$, and the st labelled trees with edge-set $E(T_{s,t}) + (a, b)$ for distinct $(a, b) \in \mathbb{Z}_s \times \mathbb{Z}_t$ are distinct labelled trees, *i.e.*, $\ell(\widetilde{T}_{s,t}) \geq st$. ■

Let the tree $T_{s,t}$ have vertices $X = \{x_1, x_2, \ldots, x_s\}$ and $Y = \{y_1, y_2, \ldots, y_t\}$, where $2 \leq s \leq t$. Tree $T_{s,t}$ has $s + t - 1$ edges so

$$s + t - 1 = \sum_{i=1}^{s} deg(x_i) = \sum_{j=1}^{t} deg(y_j). \tag{2}$$

We often use this equation for X or Y.

Leafs of $\widetilde{T}_{s,t}$, *i.e.*, vertices of degree 1, play an important role in this paper: let s_L be the number of leafs in X, so $0 \leq s_L \leq s$; and t_L the number of leafs in Y, so $0 \leq t_L \leq t$.

We next show that of the four extremal values: $s_L = 0$ or $s_L = s$ or $t_L = 0$ or $t_L = t$, only $s_L = 0$ is possible.

Lemma 2.2 *For every pair $\{s,t\}$ with $2 \le s \le t$ let $T_{s,t}$ have s_L leafs in X and t_L leafs in Y. Then:*

(i) $0 \le s_L \le s-1$, *(so $s_L = 0$ is possible);*

(ii) $1 \le t_L \le t-1$.

Proof. (i) If $s_L = s$ then every vertex $x_i \in X$ is a leaf so $deg(x_i) = 1$. Equation (2) then becomes $s+t-1 = s$, hence $t = 1$ and then $s = t = 1$, a contradiction because $s \ge 2$. Thus $0 \le s_L \le s-1$.
(ii) Now Y has t_L leafs and so $t - t_L$ vertices y_j with $deg(y_j) \ge 2$. So Equation (2) gives $s+t-1 \ge t_L + 2(t-t_L)$, and thus $t_L \ge t-s+1 \ge 1$. To show that $t_L \le t-1$ we use an argument similar to (i) starting with $t_L = t$. Thus $1 \le t_L \le t-1$. ∎

For the next Lemma we need the following result which is straightforward to prove: for $n, m \ge 0$ if $n!m! = (n+m)!$ then $n = 0$ or $m = 0$. We also need Bertrand's postulate: for every $n \ge 2$ there is a prime p for which $n < p < 2n$.

Lemma 2.3 *For every pair $\{s,t\}$ with $2 \le s \le t$, ($\{s,t\} \ne \{2,3\}$), let $\widetilde{T}_{s,t}$ satisfy $\ell(\widetilde{T}_{s,t}) = st$. Then $s_L \ge 1$, and $\widetilde{T}_{s,t}$ has at least one leaf and one non-leaf in both X and Y.*

Proof. We check from [5, 8] that the statement is true for $\{s,t\} = \{2,2\}$ and $\{3,3\}$, so assume that $t \ge 4$.

From Lemma 2.2 we have $0 \le s_L \le s-1$ and $1 \le t_L \le t-1$.

If $s_L = 0$ then Y contains all the leafs in $\widetilde{T}_{s,t}$, so $t_L \ge 2$ because any $\widetilde{T}_{s,t}$ has at least 2 leafs, and so $2 \le t_L \le t-1$. We consider two cases: (i) $2 \le t_L \le t-2$ and (ii) $t_L = t-1$.
(i) $2 \le t_L \le t-2$. As before label the vertices in X with $\mathbb{Z}_s$ and those in Y with $\mathbb{Z}_t$, and consider the st distinct labelled copies of $\widetilde{T}_{s,t}$ constructed in Lemma 2.1.

Let Y_L be the set of leafs in Y, and L be the labels on these leafs; then L is a t_L-subset of $\mathbb{Z}_t$. In each of these st copies the leafs in Y_L are labelled with translates $L+b$ of L where $b \in \mathbb{Z}_t$, and there are only t such translates. Now $2 \le t_L \le t-2$ so the number of t_L-subsets of $\mathbb{Z}_t$ is $\binom{t}{t_L} \ge \binom{t}{2} > t$. So there exists a t_L-subset L' of $\mathbb{Z}_t$ different from the t translates of L. Now take $\widetilde{T}_{s,t}$ and label the vertices in X with $\mathbb{Z}_s$, and label the leafs in Y_L with

L', and the vertices in $Y \backslash Y_L$ with $\mathbb{Z}_t \backslash L'$. Now trees whose leafs are labelled with distinct sets give distinct labelled trees, so this labelled tree is distinct from the above st labelled trees. Hence $\ell(\widetilde{T}_{s,t}) > st$, a contradiction.

(ii) $t_L = t - 1$. Here there is just one non-leaf, say y_1, in Y. From Equation (2) we have $s + t - 1 = t - 1 + deg(y_1)$. So $deg(y_1) = s$ and every vertex $x_i \in X$ is adjacent to y_1. Further, because $s_L = 0$ so $deg(x_i) \geq 2$ for all $x_i \in X$, then each x_i is also adjacent to at least one leaf in Y_L.

Now label the vertices in X with $\mathbb{Z}_s$ and those in Y with $\mathbb{Z}_t$, and label y_1 with $0_Y \in \mathbb{Z}_t$.

Let α be a non-identity permutation of X. Select $x_i \in X$ and let $\alpha(x_i) = x_{i'}$ where $i \neq i'$. Let x_i have label i_1 and $x_{i'}$ have label i_2. Let $y_j \in N(x_i) \backslash \{y_1\}$ have label j_1 and $y_{j'} \in N(x_{i'}) \backslash \{y_1\}$ have label j_2.

Now if $y_j \in N(x_{i'}) \backslash \{y_1\}$ then y_j is another non-leaf (distinct from y_1) in Y, a contradiction, so $y_j \notin N(x_{i'}) \backslash \{y_1\}$. Thus when α has been applied to X the new copy of $\widetilde{T}_{s,t}$ will have an edge with label $(i_1, j_2) \notin E(\widetilde{T}_{s,t})$; this is a new labelled copy of $\widetilde{T}_{s,t}$. By applying all $s!$ permutations of X to $\widetilde{T}_{s,t}$ we obtain $s!$ distinct labelled copies of $\widetilde{T}_{s,t}$, each with vertex y_1 labelled 0_Y. Then, by varying the label given to y_1 amongst the t distinct elements in $\mathbb{Z}_t$, we obtain $s!t$ distinct labelled copies of $\widetilde{T}_{s,t}$. Hence $\ell(\widetilde{T}_{s,t}) \geq s!t$, which is greater than st when $s \geq 3$.

For $s = 2$ let $X = \{x_1, x_2\}$ and $deg(x_1) = d_1 \geq 2$ and $deg(x_2) = d_2 \geq 2$, see Fig. 2(b); then $t = d_1 + d_2 - 1$.

If $d_1 \neq d_2$ then, from Fig. 2(b), we have $a(\widetilde{T}_{2,t}) = (d_1 - 1)!(d_2 - 1)!$. But, from Equation (1), we also have $a(\widetilde{T}_{2,t}) = \dfrac{2!t!}{2t} = (t-1)! = (d_1 + d_2 - 2)!$. Thus $(d_1 - 1)!(d_2 - 1)! = (d_1 + d_2 - 2)!$, so $d_1 = 1$ or $d_2 = 1$ by result (a) above, a contradiction.

If $d_1 = d_2 = d$ then $t = 2d - 1$. Now if $d = 2$ then $t = 3$, and so $\{s, t\} = \{2, 3\}$, a contradiction, so $d \geq 3$. Now, see Fig. 2(b), $a(\widetilde{T}_{2,2d-1}) = 2(d-1)!^2$, and, from Equation (1), we have $a(\widetilde{T}_{2,2d-1}) = (2d-2)!$, thus $2((d-1)!)^2 = (2d-2)!$. Now for $d \geq 3$, *i.e.*, for $d - 1 \geq 2$, Bertrand's postulate guarantees the existence of a prime p for which $d - 1 < p < 2d - 2$. So $p|(2d-2)!$ but $p \not| \, 2\,(d-1)!^2$, a contradiction.

Thus in both cases (i) and (ii) the assumption $s_L = 0$ leads to a contradiction, hence the result. ∎

Let P_n denote the path with n vertices, we have $a(P_n) = 2$ for $n \geq 3$. Now our first main result:

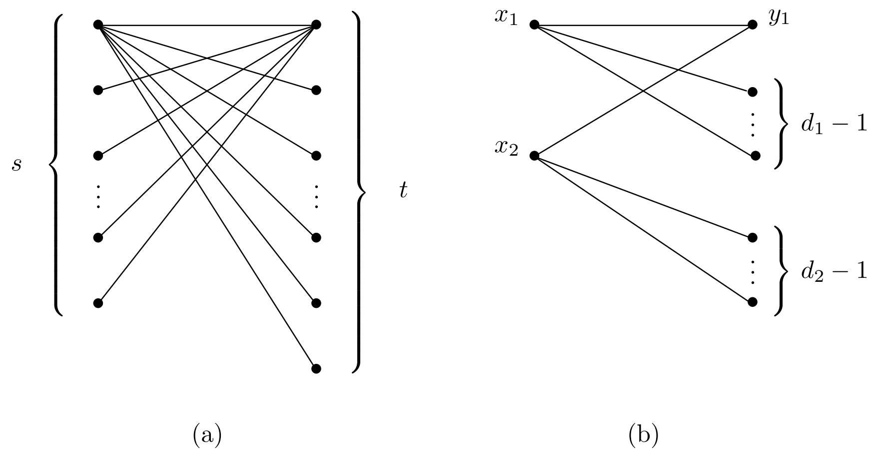


Figure 2: (a) The double-star $D_{s,t}$. (b) See proof of Lemma 2.3(ii).

Theorem 2.4 *For every pair $\{s,t\}$, with $2 \le s \le t$, ($\{s,t\} \ne \{2,3\}$), the smallest class of labelled spanning trees in $K_{s,t}$ has size st and is unique; the representative unlabelled tree is the double-star $D_{s,t}$. For $\{s,t\} = \{2,3\}$ there are two smallest classes of labelled spanning trees in $K_{2,3}$ both of size 6, one representative tree is $D_{2,3}$ and the other is the path P_5.*

Proof. For a fixed pair $\{s,t\} \ne \{2,3\}$ let $\widetilde{T}_{s,t}$ satisfy $\ell(\widetilde{T}_{s,t}) = st$ then, from Lemmas 2.2 and 2.3, we have $1 \le s_L \le s-1$ and $1 \le t_L \le t-1$, and so $\binom{s}{s_L} \ge s$ and $\binom{t}{t_L} \ge t$. As mentioned in case '(i) $2 \le t_L \le t-2$' in the proof of Lemma 2.3: trees whose leafs are labelled with distinct sets give distinct labelled trees. Hence $\widetilde{T}_{s,t}$ has at least $\binom{s}{s_L}\binom{t}{t_L}$ distinct labellings. So $\ell(\widetilde{T}_{s,t}) = st \ge \binom{s}{s_L}\binom{t}{t_L} \ge st$. Thus $\binom{s}{s_L}\binom{t}{t_L} = st$, giving four cases: (i) $(s_L, t_L) = (1,1)$, (ii) $(1, t-1)$, (iii) $(s-1, 1)$, and (iv) $(s-1, t-1)$, which we consider one-by-one:

(i) $(s_L, t_L) = (1,1)$. Let x_1 be the unique leaf in X and y_1 the unique leaf in Y. Then the $t-1$ non-leafs in $Y\backslash\{y_1\}$ and Equation (2) give $s+t-1 \ge 1 + 2(t-1)$, *i.e.*, $s \ge t$, so $s = t$ and $\ell(\widetilde{T}_{s,s}) = s^2$.

Let $x_i \in X\backslash\{x_1\}$ then $deg(x_i) \ge 2$. For a fixed $i \in \{2,3,\ldots,s\}$ Equation (2) gives $2s-1 \ge 1 + deg(x_i) + 2(s-2)$, so $deg(x_i) \le 2$, and then $deg(x_i) = 2$, *i.e.*, every vertex in $X\backslash\{x_1\}$ has degree 2; similarly every vertex in $Y\backslash\{y_1\}$ has degree 2. So $\widetilde{T}_{s,s}$ contains 2 leafs and all other vertices have degree 2, so $\widetilde{T}_{s,s} = P_{2s}$. But, using Equation (1), we have $a(P_{2s}) = 2 = \dfrac{2s!^2}{s^2} = 2(s-1)!^2$, a contradiction for $s \ge 3$. And $s = 2$ gives

$\widetilde{T}_{2,2} = P_4 = D_{2,2}$.

(ii) $(s_L, t_L) = (1, t-1)$. Here $t \geq 3$ because $t = 2$ gives case '(i) $(s_L, t_L) = (1,1)$' above. Now there is just one non-leaf, say y_1, in Y, and we use a similar argument as in case '(ii) $t_L = t-1$' in the proof of Lemma 2.3 (which works here because $s_L = 1$) to give a contradiction if $s \geq 3$. And if $s = 2$ we have $\widetilde{T}_{2,t} = D_{2,t}$ as needed.

(iii) $(s_L, t_L) = (s-1, 1)$. Here $s \geq 3$ because $s = 2$ gives case '(i) $(s_L, t_L) = (1,1)$' above, so $t \geq 3$ also. Now using the argument in case '(ii) $(s_L, t_L) = (1, t-1)$' above with s and t interchanged we obtain a contradiction when $t \geq 3$.

(iv) $(s_L, t_L) = (s-1, t-1)$. Clearly here $\widetilde{T}_{s,t} = D_{s,t}$.

Thus in each of the four cases the only solutions for $\widetilde{T}_{s,t}$ are $\widetilde{T}_{s,t} = D_{s,t}$, the double-star. This, together with analysis of the labelled trees in $K_{2,3}$, ($\{s,t\} = \{2,3\}$), gives the result. ∎

Remark The construction in the proof of Lemma 2.1 of the st labelled copies of a fixed unlabelled $\widetilde{T}_{s,t}$ gives *all* labelled copies of $\widetilde{T}_{s,t}$ when $\widetilde{T}_{s,t} = D_{s,t}$, *i.e.*, it gives the whole of the smallest class (a smallest class when $\{s,t\} = \{2,3\}$). See Fig. 3 for the tree $D_{3,4}$, also see Example 1.

As mentioned in the Introduction we also have:

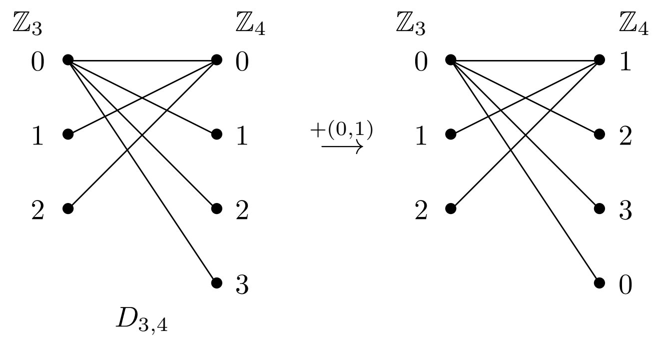
Theorem 2.5 *For every pair $\{s,t\}$ with $2 \leq s \leq t$ and for every unlabelled tree $\widetilde{T}_{s,t}$ we have $st \,|\, \ell(\widetilde{T}_{s,t})$.*

Proof. As before let all the labelled trees in a class with representative $\widetilde{T}_{s,t}$ have their vertices in X labelled with $\mathbb{Z}_s$ and in Y labelled with $\mathbb{Z}_t$, and identify such a labelled tree $T_{s,t}$ with its edge-set $E(T_{s,t})$. Now define a relation $\approx$ on this class of labelled trees:

$$E(T_{s,t}) \approx E(T'_{s,t}) \text{ iff } E(T_{s,t}) = E(T'_{s,t}) + (a,b) \text{ for some fixed } (a,b) \in \mathbb{Z}_s \times \mathbb{Z}_t.$$

Clearly $\approx$ is an equivalence relation on this class of labelled trees of size $\ell(\widetilde{T}_{s,t})$. Now all equivalence classes have size st, hence $st \,|\, \ell(\widetilde{T}_{s,t})$. ∎

Example 3 Using Theorem 2.5 we can construct all labelled copies of *any* $\widetilde{T}_{s,t}$ by first finding the st copies by the construction in the proof of Lemma 2.1, and then finding a new labelled copy outside this class if one

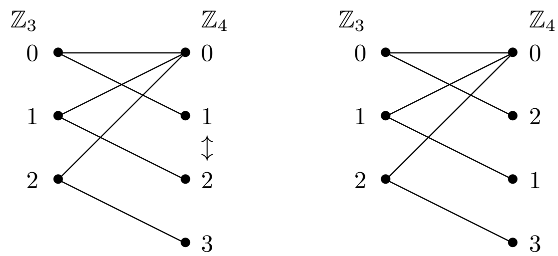


$E(D_{3,4})$ \ $+(a,b)$	$(0,0)$	$(0,1)$	$(0,2)$	$(0,3)$	$(1,0)$	$(2,0)$
$+(0,0)$	$(0,0)$	$(0,1)$	$(0,2)$	$(0,3)$	$(1,0)$	$(2,0)$
$+(0,1)$	$(0,1)$	$(0,2)$	$(0,3)$	$(0,0)$	$(1,1)$	$(2,1)$
$+(0,2)$	$(0,2)$	$(0,3)$	$(0,0)$	$(0,1)$	$(1,2)$	$(2,2)$
$+(0,3)$	$(0,3)$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,3)$	$(2,3)$
$+(1,0)$	$(1,0)$	$(1,1)$	$(1,2)$	$(1,3)$	$(2,0)$	$(0,0)$
$+(1,1)$	$(1,1)$	$(1,2)$	$(1,3)$	$(1,0)$	$(2,1)$	$(0,1)$
$+(1,2)$	$(1,2)$	$(1,3)$	$(1,0)$	$(1,1)$	$(2,2)$	$(0,2)$
$+(1,3)$	$(1,3)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,3)$	$(0,3)$
$+(2,0)$	$(2,0)$	$(2,1)$	$(2,2)$	$(2,3)$	$(0,0)$	$(1,0)$
$+(2,1)$	$(2,1)$	$(2,2)$	$(2,3)$	$(2,0)$	$(0,1)$	$(1,1)$
$+(2,2)$	$(2,2)$	$(2,3)$	$(2,0)$	$(2,1)$	$(0,2)$	$(1,2)$
$+(2,3)$	$(2,3)$	$(2,0)$	$(2,1)$	$(2,2)$	$(0,3)$	$(1,3)$

Figure 3: The edge-sets of the 12 distinct labelled copies of $D_{3,4}$ in $K_{3,4}$ with the first two copies.

exists. Then, using this as a representative for a new class, we form a further st copies, by the same additive procedure. We repeat this process until all labelled copies are found.

See Fig. 4 for the tree T24, where the 2 labelled copies which produce all 24 labelled trees are shown.



T24

E(T24) / $+(a,b)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,2)$	$(2,0)$	$(2,3)$
$+(0,0)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,2)$	$(2,0)$	$(2,3)$
$+(0,1)$	$(0,1)$	$(0,2)$	$(1,1)$	$(1,3)$	$(2,1)$	$(2,0)$
$+(0,2)$	$(0,2)$	$(0,3)$	$(1,2)$	$(1,0)$	$(2,2)$	$(2,1)$
$+(0,3)$	$(0,3)$	$(0,0)$	$(1,3)$	$(1,1)$	$(2,3)$	$(2,2)$
$+(1,0)$	$(1,0)$	$(1,1)$	$(2,0)$	$(2,2)$	$(0,0)$	$(0,3)$
$+(1,1)$	$(1,1)$	$(1,2)$	$(2,1)$	$(2,3)$	$(0,1)$	$(0,0)$
$+(1,2)$	$(1,2)$	$(1,3)$	$(2,2)$	$(2,0)$	$(0,2)$	$(0,1)$
$+(1,3)$	$(1,3)$	$(1,0)$	$(2,3)$	$(2,1)$	$(0,3)$	$(0,2)$
$+(2,0)$	$(2,0)$	$(2,1)$	$(0,0)$	$(0,2)$	$(1,0)$	$(1,3)$
$+(2,1)$	$(2,1)$	$(2,2)$	$(0,1)$	$(0,3)$	$(1,1)$	$(1,0)$
$+(2,2)$	$(2,2)$	$(2,3)$	$(0,2)$	$(0,0)$	$(1,2)$	$(1,1)$
$+(2,3)$	$(2,3)$	$(2,0)$	$(0,3)$	$(0,1)$	$(1,3)$	$(1,2)$
	$(0,0)$	$(0,2)$	$(1,0)$	$(1,1)$	$(2,0)$	$(2,3)$
$+(0,0)$	$(0,0)$	$(0,2)$	$(1,0)$	$(1,1)$	$(2,0)$	$(2,3)$
$+(0,1)$	$(0,1)$	$(0,3)$	$(1,1)$	$(1,2)$	$(2,1)$	$(2,0)$
$+(0,2)$	$(0,2)$	$(0,0)$	$(1,2)$	$(1,3)$	$(2,2)$	$(2,1)$
$+(0,3)$	$(0,3)$	$(0,1)$	$(1,3)$	$(1,0)$	$(2,3)$	$(2,2)$
$+(1,0)$	$(1,0)$	$(1,2)$	$(2,0)$	$(2,1)$	$(0,0)$	$(0,3)$
$+(1,1)$	$(1,1)$	$(1,3)$	$(2,1)$	$(2,2)$	$(0,1)$	$(0,0)$
$+(1,2)$	$(1,2)$	$(1,0)$	$(2,2)$	$(2,3)$	$(0,2)$	$(0,1)$
$+(1,3)$	$(1,3)$	$(1,1)$	$(2,3)$	$(2,0)$	$(0,3)$	$(0,2)$
$+(2,0)$	$(2,0)$	$(2,2)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,3)$
$+(2,1)$	$(2,1)$	$(2,3)$	$(0,1)$	$(0,2)$	$(1,1)$	$(1,0)$
$+(2,2)$	$(2,2)$	$(2,0)$	$(0,2)$	$(0,3)$	$(1,2)$	$(1,1)$
$+(2,3)$	$(2,3)$	$(2,1)$	$(0,3)$	$(0,0)$	$(1,3)$	$(1,2)$

Figure 4: The edge-sets of the 24 distinct labelled trees of T24 in $K_{3,4}$

Finally we consider automorphism groups:

2.1 Largest automorphism group amongst $\widetilde{T}_{s,t}$

Definition For a fixed pair $\{s,t\}$ define:

$$a_{max}(\widetilde{T}_{s,t}) = \max_{\widetilde{T}_{s,t}}\{a(\widetilde{T}_{s,t})\},$$

to be the size of the *largest* automorphism group of a $\widetilde{T}_{s,t}$ amongst all $\widetilde{T}_{s,t}$.

Now st is the smallest labelled class size for a $\widetilde{T}_{s,t}$, so Equation (1) gives:

Theorem 2.6 *For every pair $\{s,t\}$ with $2 \leq s \leq t$ we have*

$$a_{max}(\widetilde{T}_{s,t}) = \begin{cases} (s-1)!(t-1)! & s \neq t, \\ 2(s-1)!^2 & s = t. \end{cases}$$

■

We now interpret our results with respect to automorphism group size by classifying all $\widetilde{T}_{s,t}$ with the largest possible automorphism group.

Theorem 2.7 *For every pair $\{s,t\}$ with $2 \leq s \leq t$,*

(i) if $s \neq t$, $(\{s,t\} \neq \{2,3\})$, then $a(\widetilde{T}_{s,t}) = a_{max}(\widetilde{T}_{s,t}) = (s-1)!(t-1)!$ if and only if $\widetilde{T}_{s,t} = D_{s,t}$, (here $\mathrm{Aut}(D_{s,t}) = \mathrm{Sym}(s-1) \times \mathrm{Sym}(t-1)$); and if $\{s,t\} = \{2,3\}$ then $a(\widetilde{T}_{2,3}) = a_{max}(\widetilde{T}_{2,3}) = 2$ if and only if $\widetilde{T}_{2,3} = D_{2,3}$ or P_5, (here $\mathrm{Aut}(D_{2,3}) = \mathrm{Aut}(P_5) \equiv \mathbb{Z}_2$);

(ii) if $s = t$, then $a(\widetilde{T}_{s,s}) = a_{max}(\widetilde{T}_{s,s}) = 2(s-1)!^2$ if and only if $\widetilde{T}_{s,s} = D_{s,s}$, (here $\mathrm{Aut}(D_{s,s}) = \mathrm{Sym}(s-1) \boxtimes \mathrm{Sym}(2)$).

Proof.(i) Forward implication: if $\{s,t\} \neq \{2,3\}$ and $a(\widetilde{T}_{s,t}) = (s-1)!(t-1)!$ then $\ell(\widetilde{T}_{s,t}) = st$ and so, from Theorem 2.4, we have $T_{s,t} = D_{s,t}$. The backward implication is clear from the structure of $D_{s,t}$. We check the pair $\{s,t\} = \{2,3\}$ using [8]. Clearly all automorphism groups are as stated.
(ii) Similar to (i). ■

Remarks and possibilities for future research

1) In this paper we considered the smallest class of labelled spanning trees in $K_{s,t}$ and the $\widetilde{T}_{s,t}$ with largest automorphism group. In a future paper, currently in preparation, we will consider the *largest* class of labelled spanning trees in $K_{s,t}$ and the $\widetilde{T}_{s,t}$ with *smallest* automorphism group.

2) Theorem 2.5 states that: for every unlabelled tree $\widetilde{T}_{s,t}$ we have $st \mid \ell(\widetilde{T}_{s,t})$. Thus, for an unlabelled tree $\widetilde{T}_{s,t}$, we could study the parameter $\frac{\ell(\widetilde{T}_{s,t})}{st}$, and, perhaps, relate it to the structure of $\widetilde{T}_{s,t}$.

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