On k-minimum and m-minimum Edge-Magic Injections of Graphs

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Abstract

An edge-magic total labelling (EMTL) of a graph G with n vertices and e edges is an injection $\lambda\colon V(G) \cup E(G) \to [n+e]$, where, for every edge $uv \in E(G)$, we have $wt_\lambda(uv) = k_\lambda$, the magic sum of λ. An edge-magic injection (EMI) μ of G is an injection $\mu\colon V(G) \cup E(G) \to \mathbb{N}$ with magic sum k_μ and largest label m_μ. For a graph G we define and study the two parameters $\kappa(G)$: the smallest k_μ amongst all EMI's μ of G, and $\mathfrak{m}(G)$: the smallest m_μ amongst all EMI's μ of G. We find $\kappa(G)$ for $G \in \mathcal{G}$ for many classes of graphs $\mathcal{G}$. We present algorithms which compute the parameters $\kappa(G)$ and $\mathfrak{m}(G)$. These algorithms use a G-sequence: a sequence of integers on the vertices of G whose sum on edges is distinct. We find these parameters for all G with up to 7 vertices. We introduce the concept of a double-witness: an EMI μ of G for which both $k_\mu = \kappa(G)$ and $m_\mu = \mathfrak{m}(G)$; and present an algorithm to find all double-witnesses for G. The deficiency of G, $def(G)$, is $\mathfrak{m}(G) - n - e$. Two new graphs on 6 vertices with $def(G) = 1$ are presented. A previously studied parameter of G is $\kappa_{EMTL}(G)$, the magic strength of G: the smallest k_λ amongst all EMTL's λ of G. We relate $\kappa(G)$ to $\kappa_{EMTL}(G)$ for various G, and find a class of graphs $\mathcal{B}$ for which $\kappa_{EMTL}(G) - \kappa(G)$ is a constant multiple of $n-4$ for $G \in \mathcal{B}$. We specialize to $G = K_n$, and find both $\kappa(K_n)$ and $\mathfrak{m}(K_n)$ for all $n \leq 11$. We relate $\kappa(K_n)$ and $\mathfrak{m}(K_n)$ to known functions of n, and give lower bounds for $\kappa(K_n)$ and $\mathfrak{m}(K_n)$.

Keywords: edge-magic injection; magic strength; G-sequence; k-minimum G-sequence; m-minimum G-sequence; Well Spread-sequence; double-witness.

1 Introduction, $\kappa(G)$, $\mathfrak{m}(G)$

We use $\mathbb{N} = \{1, 2, 3, \ldots\}$ for the set of natural numbers, and $[s] = \{1, 2, \ldots, s\}$ for the set of the first s natural numbers. Here G will be a simple graph *without isolated vertices*, with vertex-set $V(G)$ of order $n \geq 2$, and edge-set $E(G)$ of size $e \geq 1$. Consider an injection $\mu : V(G) \cup E(G) \to \mathbb{N}$, which we represent by labelling each vertex and edge of G with a distinct natural number, this is a total labelling of G. For edge $uv \in E(G)$ let its *weight* under μ be $wt_\mu(uv) = \mu(u) + \mu(v) + \mu(uv)$.

A *magic valuation* of G is an injection $\lambda : V(G) \cup E(G) \to [n + e]$, where, for every edge $uv \in E(G)$, we have $wt_\lambda(uv) = k_\lambda$, for a constant k_λ, called the *magic sum* of λ. Magic valuations were introduced by Kotzig and Rosa in [5], and have been studied further under the name *edge-magic total labellings (EMTL's)*. Here we use 'EMTL' instead of 'magic valuation'. Wallis, Baskoro, Miller, and Slamin [14], and Wallis [13] contain much information about EMTL's. See Gallian [2] for numerous classes of graphs that have an EMTL, and for other information on this and related topics. There is now extensive ongoing research in the field of graph labellings, much of it stimulated by magic valuations which were amongst the first labellings studied.

Here we focus on *edge-magic injections* of a graph G.

Definitions: Edge-magic injection (EMI) of G; k_μ, m_μ

(1) The injection μ is an edge-magic injection of G if for every edge $uv \in E(G)$ we have $wt_\mu(uv) = k_\mu$, for some constant k_μ called the magic sum of μ.

(2) m_μ is the largest label used in μ. So $\mu \colon V(G) \cup E(G) \to [m_\mu]$.

Thus an EMI of G is a relaxed form of an EMTL of G, in that the labels of G can be any natural number. The idea of an EMI also comes from [5]. Not every graph has an EMTL, an example is K_4. But every graph has an EMI (see Theorem 3.1) and this is one advantage of studying EMI's over studying EMTL's. It appears that very little research has been carried out on EMI's as compared to EMTL's.

One avenue of research in EMTL's is to extend the list of graphs that have an EMTL. Another is to investigate properties of EMTL's. With this second idea in mind Avadayappan, Vasuki, and Jeyanthi [1] defined the following parameter, called the *magic strength* of G, for any graph G that has an EMTL. We denote this parameter by $\kappa_{EMTL}(G)$.

Definition: $\kappa_{EMTL}(G)$

Let G have an EMTL. Then

$$\kappa_{EMTL}(G) = min\{k_\lambda \,|\, \lambda \text{ is an EMTL of } G \text{ with magic sum } k_\lambda\},$$

is the smallest k_λ amongst all EMTL's λ of G.

All known values of $\kappa_{EMTL}(G)$ are given in Theorem 1.1 below.

In [6] it was shown that K_n has an EMTL if and only if $n \in \{2, 3, 5, 6\}$. The values of $\kappa_{EMTL}(K_n)$ for $n \in \{2, 3, 5, 6\}$ in line 1 of Theorem 1.1 come from Section 7.1 of [14] where all such EMTL's were found, although $\kappa_{EMTL}(G)$ was not considered; see also Section 2.3.3 of [13]. The values of $\kappa_{EMTL}(G)$ for the graphs G in lines 2–6 of Theorem 1.1 are from [1], and those in lines 7 and 8 are from Section 2 of Murugan [7].

Remark: All graphs G in Theorem 1.1 that contain an 'n' in their notation have n vertices, except for $G = B^*_{\frac{n-2}{2},\frac{n-2}{2}}$ which has $n + 1$ vertices.

K_n, P_n, and C_n represent the complete graph, the path, and the cycle respectively. The bi-star $B_{\frac{n-2}{2},\frac{n-2}{2}}$ for n even and ≥ 4 is obtained from two disjoint copies of the star $K_{1,\frac{n-2}{2}}$ by joining the center vertices with a new edge; and $B^*_{\frac{n-2}{2},\frac{n-2}{2}}$ is obtained from $B_{\frac{n-2}{2},\frac{n-2}{2}}$ by subdividing this new edge with a new vertex, it has $n + 1$ vertices. The Huffman tree $HT_{\frac{n+1}{2}}$ for n odd and ≥ 3 is the path $P_{\frac{n+1}{2}}$ with a pendant edge attached to every vertex except the last. The twig $TW_{\frac{n}{3}}$ for $n \equiv 3 \pmod 6$ is the path $P_{\frac{n}{3}}$ with two pendant edges attached to every vertex.

Theorem 1.1 ([14], [1], [7])

$$\kappa_{EMTL}(G) = \begin{cases} 6,\ 9,\ 18,\ 25, & G = K_2,\ K_3,\ K_5,\ K_6; \\ \left\lceil \dfrac{5n+3}{2} \right\rceil, & G = C_n, (n \geq 3); \\ \dfrac{9n+6}{4}, & G = \frac{n}{2}K_2, (n \equiv 2\ (mod\ 4)); \\ \left\lceil \dfrac{5n+1}{2} \right\rceil, & G = P_n, (n \geq 2); \\ \dfrac{5n+2}{2}, & G = B_{\frac{n-2}{2},\frac{n-2}{2}}, (n\ even\ and \geq 4); \\ 2n+5, & G = B^*_{\frac{n-2}{2},\frac{n-2}{2}}, (n\ even\ and \geq 4); \\ \left\lceil \dfrac{9n+5}{4} \right\rceil, & G = HT_{\frac{n+1}{2}}, (n\ odd\ and \geq 3); \\ \dfrac{13n+9}{6}, & G = TW_{\frac{n}{3}}, (n \equiv 3\ (mod\ 6)). \end{cases}$$

■

Because every graph G has an EMI, we may define a new parameter, $\kappa(G) = \kappa_{EMI}(G)$, as a more general version of $\kappa_{EMTL}(G)$. It appears that $\kappa(G)$ has not been considered before.

Definitions: $\kappa(G)$, k-minimum EMI of G; witness for $\kappa(G)$

(1) $\kappa(G) = min\{k_\mu \,|\, \mu$ is an EMI of G with magic sum $k_\mu\}$, is the smallest k_μ amongst all EMI's μ of G.

(2) EMI μ is a k-minimum EMI of G if $k_\mu = \kappa(G)$; and μ is a witness for $\kappa(G)$.

See Sections 3 and 4 of [7] for related, but different, parameters of G; and see Kong, Lee, and Sun [3] for a similar parameter, but concerning the vertices of G.

The second parameter of a graph G which we consider is $\mathfrak{m}(G)$.

Definitions: $\mathfrak{m}(G)$, m-minimum EMI of G; witness for $\mathfrak{m}(G)$, $def(G)$

(1) $\mathfrak{m}(G) = min\{m_\mu \,|\, \mu$ is an EMI of G with largest label $m_\mu\}$, is the smallest m_μ amongst all EMI's μ of G.

(2) EMI μ is a m-minimum EMI of G if $m_\mu = \mathfrak{m}(G)$; and μ is a witness for $\mathfrak{m}(G)$.

(3) $def(G) = \mathfrak{m}(G) - n - e$, the deficiency of G, is the smallest number such that there exists an EMI $\mu\colon V(G) \cup E(G) \to [n + e + def(G)]$.

The concept of 'deficiency' comes from [5]; the formulation we use is slightly different from that used there. By Definition (1) above it is clear that $\mathfrak{m}(G) = n+e$ if and only if G has an EMTL (if and only if $def(G) = 0$). As an example of some graphs G with $\mathfrak{m}(G) = n + e + 1$ (equivalently, $def(G) = 1$) see the 10 graphs shown in [5], each has $n \leq 6$.

We summarize our paper:

In Section 2 we give a lower bound for $\kappa(G)$, and then find $\kappa(G)$ for all incomplete graphs G in Theorem 1.1. In Section 3 we define a G-sequence A, and show that every graph G on n vertices has an EMI. In Section 4 we present algorithms that, for a fixed G, compute $\kappa(G)$, and find all witnesses for $\kappa(G)$, *i.e.*, all k-minimum EMI's of G. In Section 5 we present algorithms that compute $\mathfrak{m}(G)$, and find all witnesses for $\mathfrak{m}(G)$, *i.e.*, find all m-minimum EMI's of G. In Section 6 we present our results from the algorithms of Sections 4 and 5 for graphs G with $n = 2, 3, 4, 5$, or 6 vertices. We find two new graphs G with $def(G) = 1$ on 6 vertices. We also consider graphs G with $n = 7$ vertices, and trees T with up to $n = 10$ vertices. In Section 7 we specialize to $G = K_n$. Finally, in Section 8, we consider miscellaneous items.

2 $\kappa(G)$ for certain G

In this section we find $\kappa(G)$ for all graphs G in Theorem 1.1 except for $G = K_n$ where $n \in \{2, 3, 5, 6\}$. In Section 7 we show $\kappa(K_n) = \kappa_{EMTL}(K_n)$ for all $n \in \{2, 3, 5, 6\}$, (see Table 3).

If a graph G has an EMTL then, since an EMTL of G is an EMI of G, we have

$$\kappa(G) \leq \kappa_{EMTL}(G). \tag{1}$$

However $\kappa(G) < \kappa_{EMTL}(G)$ is possible. From [1] the graph $B_{2,2}$ has $\kappa_{EMTL}(B_{2,2}) = 16$, but see Fig. 1 for an EMI μ of $B_{2,2}$ with $k_\mu = 15 < 16$. Indeed, in Theorem 2.4(ii), we show that $\kappa(B_{2,2}) = 15$.

In Theorem 2.1 below we let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let d_i denote the degree of vertex v_i for $i = 1, 2, \ldots, n$. We order the vertices so that $d_1 \leq d_2 \leq \cdots \leq d_n$.

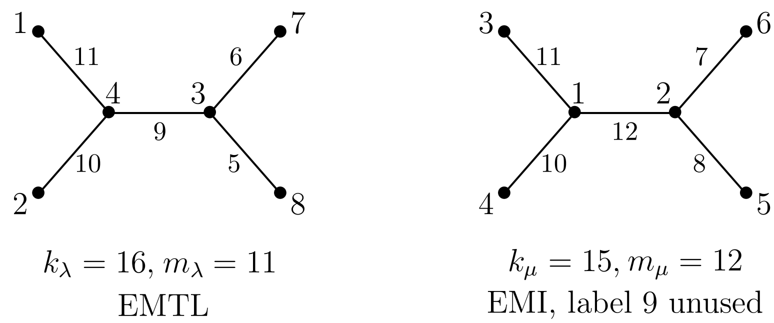


Figure 1: A k-minimum EMTL and a k-minimum EMI of $B_{2,2}$

Theorem 2.1 *Let G have n vertices, e edges, and vertex degrees $[d_1, d_2, \ldots, d_n]$ where $d_1 \le d_2 \le \cdots \le d_n$. Then*

(i) $\kappa(G) \ge \left\lceil \dfrac{6ne + e^2 + 3e + 2\mathfrak{m}(G) - 2n - 2\sum_{i=1}^{n} i\, d_i}{2e} \right\rceil$,

(ii) $\kappa(G) \ge \left\lceil \dfrac{6ne + e^2 + 5e - 2\sum_{i=1}^{n} i\, d_i}{2e} \right\rceil$.

Proof. (i) Let μ be an EMI of G with magic sum $\kappa(G)$ and with largest label m_μ. When summing $\kappa(G)$ over every edge, each edge label $\mu(uv)$ appears one time and each vertex label $\mu(v_i)$ appears d_i times. That is,

$$e\kappa(G) = \sum_{uv \in E(G)} \mu(uv) + \sum_{v_i \in V(G)} d_i\, \mu(v_i). \tag{2}$$

In order to minimize the RHS of Equation (2) we use the $n+e$ labels $\{1, 2, \ldots, n+e-1, m_\mu\}$ on G. We place the largest e labels $\{n+1, n+2, \ldots, n+e-1, m_\mu\}$ on the e edges; and the smallest n labels $\{1, 2, \ldots, n\}$ on the n vertices in reverse order, so that $\mu(v_i) = n-i+1$ for $i = 1, 2, \ldots, n$. So

$$e\kappa(G) \ge \frac{2ne + 2m_\mu + e^2 - 2n - e}{2} + \sum_{i=1}^{n} d_i\,(n-i+1).$$

Now, using $\sum_{i=1}^{n} d_i = 2e$, and $m_\mu \ge \mathfrak{m}(G)$, and noting that $\kappa(G)$ is an integer, gives the result.

(ii) Use (i) and $\mathfrak{m}(G) \geq n + e$. ■

Corollary 2.2 *Let G be a r-regular graph with n vertices and e edges. Then*

(i) $\kappa(G) \geq \left\lceil \dfrac{4ne + e^2 + e + 2\mathfrak{m}(G) - 2n}{2e} \right\rceil$,

(ii) $\kappa(G) \geq \left\lceil \dfrac{4n + e + 3}{2} \right\rceil$.

Proof. For (i) we use Theorem 2.1, and $nr = 2e$, and $\sum_{i=1}^{n} i = \binom{n+1}{2}$. And for (ii) we use (i) and $\mathfrak{m}(G) \geq n + e$. ■

The graphs G in lines 2 and 3 in Theorem 1.1 are regular, we have:

Theorem 2.3

(i) $\kappa(C_n) = \kappa_{EMTL}(C_n) = \left\lceil \dfrac{5n+3}{2} \right\rceil, (n \geq 3)$.

(ii) $\kappa(\frac{n}{2}K_2) = \kappa_{EMTL}(\frac{n}{2}K_2) = \dfrac{9n+6}{4}, (n \equiv 2\ (mod\ 4))$.

Proof. (i) When $G = C_n$ we have $e = n$, and Corollary 2.2(ii) gives $\kappa(C_n) \geq \lceil \frac{5n+3}{2} \rceil = \kappa_{EMTL}(C_n)$. The upperbound $\kappa(C_n) \leq \kappa_{EMTL}(C_n)$ comes from Equation (1). The proof of (ii) is similar using $e = \frac{n}{2}$. ■

Now for the irregular G in lines 4–8 in Theorem 1.1 we have:

Theorem 2.4

(i) $\kappa(P_n) = \kappa_{EMTL}(P_n) = \left\lceil \dfrac{5n+1}{2} \right\rceil, (n \geq 2).$

(ii) $\kappa(B_{\frac{n-2}{2},\frac{n-2}{2}}) = 2n+3, (n \textit{ even and } \geq 4).$

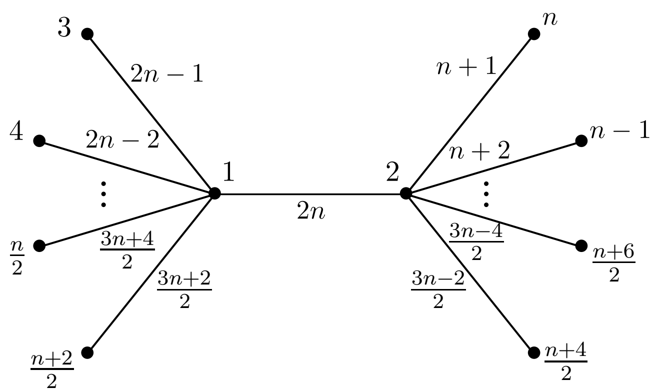
(iii) $\kappa(B^*_{\frac{n-2}{2},\frac{n-2}{2}}) = \kappa_{EMTL}(B^*_{\frac{n-2}{2},\frac{n-2}{2}}) = 2n+5, (n \textit{ even and } \geq 4).$

(iv) $\kappa(HT_{\frac{n+1}{2}}) = \kappa_{EMTL}(HT_{\frac{n+1}{2}}) = \left\lceil \dfrac{9n+5}{4} \right\rceil, (n \textit{ odd and } \geq 3).$

(v) $\kappa(TW_{\frac{n}{3}}) = \kappa_{EMTL}(TW_{\frac{n}{3}}) = \dfrac{13n+9}{6}, (n \equiv 3 \ (mod\ 6)).$

Proof. (i) Here $G = P_n$ with n vertices and $e = n-1$ edges and degrees $[1, 1, \overbrace{2, \ldots, 2}^{n-2}]$, so $\sum id_i = n^2 + n - 3$. Then Theorem 2.1(ii) gives $\kappa(P_n) \geq \left\lceil \frac{5n}{2} + \frac{1}{n-1} \right\rceil = \left\lceil \frac{5n+1}{2} \right\rceil$. And $\kappa(P_n) \leq \kappa_{EMTL}(P_n) = \left\lceil \frac{5n+1}{2} \right\rceil$ comes from Theorem 1.1 and Equation (1). Thus $\kappa(P_n) = \kappa_{EMTL}(P_n) = \left\lceil \frac{5n+1}{2} \right\rceil$.

(ii) Here $G = B_{\frac{n-2}{2},\frac{n-2}{2}}$ with n vertices and $e = n-1$ edges and degrees $[\overbrace{1, \ldots, 1}^{n-2}, \frac{n}{2}, \frac{n}{2}]$, so $\sum id_i = \frac{3n^2-4n+2}{2}$. Then Theorem 2.1(ii) gives $\kappa(B_{\frac{n-2}{2},\frac{n-2}{2}}) \geq \left\lceil \frac{4n+5}{2} - \frac{1}{2(n-2)} \right\rceil = 2n+3$. Now consider the EMI μ of $B_{\frac{n-2}{2},\frac{n-2}{2}}$ shown below. It has $k_\mu = 2n+3$ and $m_\mu = 2n$, label $\frac{3n}{2}$ is unused. Hence $\kappa(B_{\frac{n-2}{2},\frac{n-2}{2}}) \leq 2n+3$, and so $\kappa(B_{\frac{n-2}{2},\frac{n-2}{2}}) = 2n+3$.



(iii) Here $G = B^*_{\frac{n-2}{2},\frac{n-2}{2}}$ with $n+1$ vertices and $e = n$ edges and degrees $[\overbrace{1,\ldots,1}^{n-2}, 2, \frac{n}{2}, \frac{n}{2}]$, so $\sum id_i = \frac{3n^2+2n-2}{2}$. Theorem 2.1(ii) gives $\kappa(B^*_{\frac{n-2}{2},\frac{n-2}{2}}) \geq \left\lceil \frac{4n+9}{2} + \frac{1}{n} \right\rceil = 2n+5$. And we have $\kappa(B^*_{\frac{n-2}{2},\frac{n-2}{2}}) \leq 2n+5$ from Theorem 1.1 and Equation (1). Thus $\kappa(B^*_{\frac{n-2}{2},\frac{n-2}{2}}) = \kappa_{EMTL}(B^*_{\frac{n-2}{2},\frac{n-2}{2}}) = 2n+5$.
(iv) Here $G = HT_{\frac{n+1}{2}}$ with n vertices and $e = n-1$ edges and degrees $[\overbrace{1,\ldots,1}^{\frac{n+1}{2}}, 2, \overbrace{3,\ldots,3}^{\frac{n-3}{2}}]$, so $\sum id_i = \frac{5n^2-9}{4}$. Again Theorem 2.1(ii) gives $\kappa(HT_{\frac{n+1}{2}}) \geq \left\lceil \frac{9n+3}{4} + \frac{1}{2n-2} \right\rceil = \left\lceil \frac{9n+5}{4} \right\rceil$, and $\kappa(HT_{\frac{n+1}{2}}) \leq \left\lceil \frac{9n+5}{4} \right\rceil$ from Theorem 1.1 and

Equation (1). Thus $\kappa(HT_{\frac{n+1}{2}}) = \kappa_{EMTL}(HT_{\frac{n+1}{2}}) = \lceil \frac{9n+5}{4} \rceil$.
(v) Here $G = TW_{\frac{n}{3}}$ with n vertices and $e = n-1$ edges and degrees $[\overbrace{1,\ldots,1}^{\frac{2n}{3}}, 3, 3, \overbrace{4,\ldots,4}^{\frac{n-6}{3}}]$, so $\sum id_i = \frac{4n^2-n-9}{3}$. Theorem 2.1(ii) gives $\kappa(TW_{\frac{n}{3}}) \geq \lceil \frac{13n+6}{6} + \frac{2}{n-1} \rceil = \frac{13n+9}{6}$, and $\kappa(TW_{\frac{n}{3}}) \leq \frac{13n+9}{6}$ from Theorem 1.1 and Equation (1). Thus $\kappa(TW_{\frac{n}{3}}) = \kappa_{EMTL}(TW_{\frac{n}{3}}) = \frac{13n+9}{6}$. ■

Remark: See Figure 1 for an example of the EMI of Theorem 2.4(ii) with $n = 6$. We have $\kappa_{EMTL}(B_{\frac{n-2}{2},\frac{n-2}{2}}) - \kappa(B_{\frac{n-2}{2},\frac{n-2}{2}}) = \frac{n-4}{2}$. Hence, for a graph G with n vertices, we can have the difference $\kappa_{EMTL}(G) - \kappa(G)$ as large as a constant multiple of $n-4$, *i.e.*, a constant multiple of n for sufficiently large n.

We conclude this section with the following result which could be useful when searching for an EMI μ of G with $k_\mu < \kappa_{EMTL}(G)$ for regular G with an EMTL.

Theorem 2.5 *Let G be a r-regular graph ($r \geq 2$) with an EMTL, and let λ be a witness for $\kappa_{EMTL}(G)$, so $k_\lambda = \kappa_{EMTL}(G)$. Now suppose that μ is an EMI of G with $k_\mu < \kappa_{EMTL}(G)$. Then*

$$\sum_{u \in V(G)} \mu(u) < \sum_{u \in V(G)} \lambda(u).$$

Proof. Let $S_\lambda = \sum_{u \in V(G)} \lambda(u) + \sum_{uv \in E(G)} \lambda(uv)$ be the sum of all the labels of λ, define S_μ similarly.

Now λ is an EMTL of G with $k_\lambda = \kappa_{EMTL}(G)$, so

$$e\kappa_{EMTL}(G) = \sum_{u \in V(G)} r\lambda(u) + \sum_{uv \in E(G)} \lambda(uv) = (r-1) \sum_{u \in V(G)} \lambda(u) + S_\lambda.$$

The first equality is true since each vertex label $\lambda(u)$ appears r times and each edge label $\lambda(uv)$ appears one time when summing $\kappa_{EMTL}(G)$ over every edge. Similarly,

$$ek_\mu = (r-1) \sum_{u \in V(G)} \mu(u) + S_\mu.$$

Now $k_\mu < \kappa_{EMTL}(G)$ so μ is not an EMTL of G, and thus $S_\lambda < S_\mu$. So

$$e\kappa_{EMTL}(G) - (r-1) \sum_{u \in V(G)} \lambda(u) < ek_\mu - (r-1) \sum_{u \in V(G)} \mu(u).$$

So
$$(r-1)\left(\sum_{u\in V(G)}\mu(u)-\sum_{u\in V(G)}\lambda(u)\right)<e\left(k_\mu-\kappa_{EMTL}(G)\right)<0.$$
The last inequality is true since $k_\mu < \kappa_{EMTL}(G)$. But $r \geq 2$, and so
$$\sum_{u\in V(G)}\mu(u)<\sum_{u\in V(G)}\lambda(u).$$
■

3 G-sequences

Let S and T be sets of *distinct* natural numbers, ordered or unordered.
Definitions: $S\!\uparrow$, $S\!\downarrow$, $S+T$

(1) $S\!\uparrow = max\{s \mid s \in S\}$, is the largest element in S.

(2) $S\!\downarrow = min\{s \mid s \in S\}$, is the smallest element in S.

(3) $S+T=\{s+t \mid s\in S, t\in T\}$.

Let G have vertex set $V(G)=(v_1,v_2,\ldots,v_n)$ for $n\geq 2$, fixed in this order. Let $A=(a_1,a_2,\ldots,a_n)$ be an ordered sequence of n distinct natural numbers. Now label vertex v_i with a_i, say $\ell(v_i)=a_i$, for each $i=1,2,\ldots,n$.
Definitions: $\mathcal{P}(A)$, G-sequence, $G(A)'$, $k(G(A))$, $\mathcal{E}(A)$, $G(A)$, $m(G(A))$

(1) $\mathcal{P}(A)=\{a_i+a_j \mid v_iv_j\in E(G), 1\leq i<j\leq n\}$.

(2) A is a G-sequence if each $a_i+a_j\in\mathcal{P}(A)$ is distinct, equivalently, if $|\mathcal{P}(A)|=e=|E(G)|$.

(3) $G(A)'$ is the graph G whose vertices have been labelled with elements of A.

(4) $k(G(A))$ is the smallest integer $\geq \mathcal{P}(A)\!\uparrow+1$ that lies outside $A+\mathcal{P}(A)$. That is, $k(G(A))=min\{[\mathcal{P}(A)\!\uparrow+1,\infty)\cap\overline{(A+\mathcal{P}(A))}\}$.

(5) $\mathcal{E}(A)=\{k(G(A))-a_i-a_j \mid v_iv_j\in E(G), 1\leq i<j\leq n\}\subset\mathbb{N}$, are the edge labels of G.

(6) $G(A)$ is the total labelling of G with vertex labels A and edge labels $\mathcal{E}(A)$: for edge $v_iv_j \in E(G)$ where $1 \le i < j \le n$ let its label be $\ell(v_iv_j) = k(G(A)) - a_i - a_j \in \mathcal{E}(A)$.

(7) $m(G(A)) = max\{A\uparrow, k(G(A)) - \mathcal{P}(A)\downarrow\}$, is the maximum of the vertex labels and the edge labels on $G(A)$.

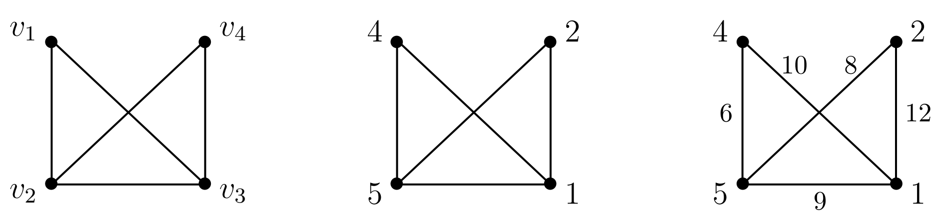
Theorem 3.1 *$G(A)$ is an EMI of G with magic sum $k(G(A))$.*

Proof. The vertex labels of $G(A)$ are distinct, and, since A is a G-sequence, then each edge label $k(G(A)) - a_i - a_j$ is also distinct. Furthermore, since $k(G(A)) \notin A + \mathcal{P}(A)$, then every $k(G(A)) - a_i - a_j \notin A$, *i.e.*, every edge label is different from every vertex label. Thus this total labelling of G is an injection into $\mathbb{N}$. It has magic sum $k(G(A))$, so is an EMI. ∎

Example 1

Consider the graph G shown: $A = (4, 5, 1, 2)$ is a G-sequence since all numbers in $\mathcal{P}(A) = \{3, 5, 6, 7, 9\}$ are distinct. Then $\mathcal{P}(A)\uparrow = 9$ and $A + \mathcal{P}(A) = \{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$. So $k(G(A)) = min\{[10, \infty) \cap (\{1, 2, 3\} \cup [15, \infty\})$=15. This gives $\mathcal{E}(A) = \{6, 8, 9, 10, 12\}$, the edge labels on $G(A)$. Finally $A\uparrow = 5$ and $\mathcal{P}(A)\downarrow = 3$, so $m(G(A)) = max\{5, 15 - 3\} = 12$ is the largest label on $G(A)$.

G $\qquad$ $G(A)'$ $\qquad$ $G(A)$



$V(G) = (v_1, v_2, v_3, v_4)$ $\qquad$ $A = (4, 5, 1, 2)$ $\qquad$ $A = (4, 5, 1, 2)$, $\mathcal{E}(A) = \{6, 8, 9, 10, 12\}$

4 Computing $\kappa(G)$, k-minimum EMI's of G

In this section G is a fixed graph. We present an algorithm to compute $\kappa(G)$, and a second algorithm to compute all witnesses for $\kappa(G)$.

Definition: $k(G)$
$k(G) = min\{k(G(A)) \,|\, A$ is a G-sequence$\}$, is the smallest value of $k(G(A))$ amongst all G-sequences A.

Theorem 4.1 *We have $\kappa(G) = k(G)$.*

Proof. To see that $\kappa(G) \leq k(G)$ let $X = (x_1, x_2, \ldots, x_n)$ be a G-sequence with $k(G(X)) = k(G)$. By Theorem 3.1 the EMI $G(X)$ of G has magic sum $k(G(X)) = k(G)$. Hence, by definition of $\kappa(G)$, we have $\kappa(G) \leq k(G)$.

Conversely let μ be a k-minimum EMI of G, so $k_\mu = \kappa(G)$; and let $A = (a_1, a_2, \ldots, a_n)$ be the sequence of vertex labels of this G written in the same order as $V(G) = (v_1, v_2, \ldots, v_n)$. Then $\kappa(G) - a_i - a_j$ for all pairs $\{i, j\}$ where $v_i v_j \in E(G)$ and $1 \leq i < j \leq n$ are the edge labels of this G. Since μ is an EMI then these are all distinct, hence the $a_i + a_j$ are distinct, and so A is a G-sequence. The smallest edge label can be 1 so, $\kappa(G) - \mathcal{P}(A)\uparrow \geq 1$, *i.e.*, $\kappa(G) \geq \mathcal{P}(A)\uparrow +1$. Also, if $\kappa(G) \in A + \mathcal{P}(A)$ then $\kappa(G) - a_i - a_j \in A$ for some $v_i v_j \in E(G)$ with $1 \leq i < j \leq n$, thus some edge label is equal to a vertex label, a contradiction. So $\kappa(G) \notin A + \mathcal{P}(A)$. Thus $\kappa(G)$ is an integer $\geq \mathcal{P}(A)\uparrow +1$ that lies outside $A + \mathcal{P}(A)$, but $k(G(A))$ is the smallest such integer. Hence $\kappa(G) \geq k(G(A)) \geq k(G)$, since A is a G-sequence.

Combining the above paragraphs gives $\kappa(G) = k(G)$. ∎

So to compute $\kappa(G)$ we will compute $k(G)$, see Algorithm $\kappa(G)$ below.

Definition: $\mathcal{W}_k(A)$
Let $A = (a_1, a_2, \ldots, a_n)$ be a G-sequence.

$$\mathcal{W}_k(A) = \{W \,|\, W \textit{ is a } G-\textit{sequence with } \mathcal{P}(W)\uparrow +1 \leq k(G(A))\}.$$

Note that $A \in \mathcal{W}_k(A)$. Note also that if $W \in \mathcal{W}_k(A)$ then $W\uparrow \leq \left\lceil \frac{k(G(A))-1}{2} \right\rceil$ so $|\mathcal{W}_k(A)| \leq Perm\left(n, \left\lceil \frac{k(G(A))-1}{2} \right\rceil\right)$, the number of permutations of length n from the set $\left[\left\lceil \frac{k(G(A))-1}{2} \right\rceil\right]$; *i.e.*, $|\mathcal{W}_k(A)|$ is finite.

Theorem 4.2 *Let A be a G-sequence. Then*

$$k(G) = min\{k(G(W)) \,|\, W \in \mathcal{W}_k(A)\}.$$

Proof. Let X be a G-sequence with $k(G(X)) = k(G)$. If $k(G) \neq min\{k(G(W)) \,|\, W \in \mathcal{W}_k(A)\}$ then $X \notin \mathcal{W}_k(A)$, *i.e.*, $\mathcal{P}(X)\!\uparrow +1 > k(G(A))$. So $k(G(X)) \geq \mathcal{P}(X)\!\uparrow +1 > k(G(A))$, *i.e.*, $k(G) > k(G(A))$, a contradiction to the minimality of $k(G)$. Hence the result. ■

The following algorithm is a finite procedure for computing $k(G)$, *i.e.*, $\kappa(G)$, it uses Theorem 4.2.

Algorithm κ(G): Compute $\kappa(G)$

(1) `Let` $A_0 = (a_1, a_2, \ldots, a_n)$ `be a` G`-sequence.`

(2) `Compute` $k(G(A_0))$.

(3) `List` $\mathcal{W}_k(A_0)$ `in lexicographic order.`

(4) `For each` $W \in \mathcal{W}_k(A_0)$ `compute` $k(G(W))$:

`IF we find` $W = A_1$ `with` $k(G(A_1)) < k(G(A_0))$

`THEN let` $A_0 = A_1$ `at Step (2) and repeat`

`ELSE output` $\kappa(G) = k(G) = k(G(A_0))$.

Definition: k-minimum G-sequence; witness for $\kappa(G)$

Let $X = (x_1, x_2, \ldots, x_n)$ be a G-sequence. Then X is a k-minimum G-sequence if $k(G(X)) = \kappa(G)$; we also call X a witness for $\kappa(G)$.

Remark: So a 'witness for $\kappa(G)$' can be both a G-sequence X or the corresponding EMI $G(X)$ of G. A Corollary of Theorem 4.1 is then:

Corollary 4.3 *G-sequence X is a witness for $\kappa(G)$ if and only if EMI $G(X)$ is a witness for $\kappa(G)$.* ■

Once $\kappa(G)$ is known we can find *all* k-minimum G-sequences, *i.e.*, all witnesses for $\kappa(G)$.

Theorem 4.4 *Let X be a k-minimum G-sequence. Then all k-minimum G-sequences lie in $\mathcal{W}_k(X)$.*

Proof. Let W be a k-minimum G-sequence, then $k(G(W)) = \kappa(G)$. So $\mathcal{P}(W)\!\uparrow +1 \leq k(G(W)) = \kappa(G) = k(G(X))$, *i.e.*, $W \in \mathcal{W}_k(X)$.

■

We give an algorithm to find all witnesses W for $\kappa(G)$. For this we need:

Theorem 4.5

(i) Let μ be a k-minimum EMI of G. Then $1 \in \mathbb{N}$ appears as a label of G.

(ii) Let $X = (x_1, x_2, \ldots, x_n)$ be a k-minimum G-sequence. Then either $x_i = 1$ for some $x_i \in X$, or $\mathcal{P}(X)\!\uparrow +1 = \kappa(G)$, (not both).

Proof. (i) Let $\ell \in \mathbb{N}$ be the smallest label used in μ. Now define an injection $\mu' : V(G) \cup E(G) \to \mathbb{N}$ given by $\mu'(w) = \mu(w) - (\ell - 1)$ for all $w \in V(G) \cup E(G)$. It is straightforward to check that, if $\ell > 1$, then μ' is an EMI of G with magic sum $k_{\mu'} = k_\mu - 3(\ell - 1) < k_\mu = \kappa(G)$, a contradiction. Hence $\ell = 1$ as required.
(ii) Since $k(G(X)) = \kappa(G)$ then the labelled graph $G(X)$ is a k-minimum EMI of G, and so, from (i), the label 1 has been used on a vertex or an edge, (not both). If 1 is a vertex label on some v_i then $x_i = 1$. Or, if 1 is an edge label, then it is the smallest edge label, so $1 = k(G(X)) - \mathcal{P}(X)\!\uparrow = \kappa(G) - \mathcal{P}(X)\!\uparrow$. That is, $\mathcal{P}(X)\!\uparrow +1 = \kappa(G)$. ∎

Remark: See Table 3, $n = 2$ where both cases of Theorem 4.5(ii) are illustrated.

Definition: $witness(\kappa(G))$

$$witness(\kappa(G)) = \{W \mid W \textit{ is a } k\textit{-minimum } G\textit{-sequence}\}.$$

We use Theorem 4.5(ii) in the following algorithm where we assume that $\kappa(G)$ is known.

Algorithm witness(κ(G)): Find $witness(\kappa(G))$

(1) `Let` $X = (x_1, x_2, \ldots, x_n)$ `be a` k`-minimum` G`-sequence.`

(2) `List all` $W = (w_1, w_2, \ldots, w_n) \in \mathcal{W}_k(X)$ `with some` $w_i = 1$ `or` $\mathcal{P}(W)\!\uparrow +1 = \kappa(G)$`, (not both).`

(3) `For these` W `compute` $k(G(W))$`.`

`IF` $k(G(W)) = \kappa(G)$

`THEN output` W `into` $witness(\kappa(G))$

`ELSE reject` W`.`

5 Computing $\mathfrak{m}(G)$, m-minimum EMI's of G

In this section G is fixed. We present an algorithm to compute $\mathfrak{m}(G)$, and a second algorithm to compute all witnesses for $\mathfrak{m}(G)$.

Recall Definition (7) of Section 3: For a G-sequence $A = (a_1, a_2, \ldots, a_n)$ we have $m(G(A)) = max\{A\uparrow, k(G(A)) - \mathcal{P}(A)\downarrow\}$. So $m(G(A))$ is the maximum of the vertex labels and the edge labels of $G(A)$, *i.e.*, the largest label of $G(A)$. Analogous to the definition of $k(G)$:

Definition: $m(G)$
$m(G) = min\{m(G(A)) \,|\, A \ is \ a \ G-sequence\}$, is the smallest value of $m(G(A))$ amongst all G-sequences A.

Analogous to Theorem 4.1:

Theorem 5.1 *We have* $\mathfrak{m}(G) = m(G)$.

Proof. To see that $\mathfrak{m}(G) \leq m(G)$ let $Y = (y_1, y_2, \ldots, y_n)$ be a G-sequence with $m(G(Y)) = m(G)$. Now the total labelling $G(Y)$ of G is an EMI μ of G with $m_\mu = m(G(Y)) = m(G)$. Hence, by definition of $\mathfrak{m}(G)$, we have $\mathfrak{m}(G) \leq m(G)$.

Conversely let μ be a m-minimum EMI of G, so $m_\mu = \mathfrak{m}(G)$; and let $A = (a_1, a_2, \ldots, a_n)$ be the sequence of vertex labels of this G written in the same order as $V(G) = (v_1, v_2, \ldots, v_n)$. Then $k_\mu - a_i - a_j$ are the edge labels of this G for edges $v_i v_j$ where $1 \leq i < j \leq n$. As before, these are all distinct, hence the $a_i + a_j$ are distinct, and A is a G-sequence. Now by definition of $k(G(A))$ we have $k(G(A)) \leq k_\mu$. So $m(G(A)) = max\{A\uparrow, k(G(A)) - \mathcal{P}(A)\downarrow\} \leq max\{A\uparrow, k_\mu - \mathcal{P}(A)\downarrow\} = m_\mu = \mathfrak{m}(G)$. And then by definition of $m(G)$, since A is a G-sequence, we have $m(G) \leq m(G(A)) \leq \mathfrak{m}(G)$, as needed.

Combining the above paragraphs gives $\mathfrak{m}(G) = m(G)$. ∎

So, to compute $\mathfrak{m}(G)$ we will compute $m(G)$, see Algorithm $\mathfrak{m}(G)$ below.

Definition: $\mathcal{W}_m(A)$
Let $A = (a_1, a_2, \ldots, a_n)$ be a G-sequence.

$$\mathcal{W}_m(A) = \{W \,|\, W \ is \ a \ G-sequence \ with \ W\uparrow \leq m(G(A))\}.$$

Note that $A \in \mathcal{W}_m(A)$. Note also that $|\mathcal{W}_m(A)| \leq Perm\,(n, m(G(A)))$, *i.e.*, $|\mathcal{W}_m(A)|$ is finite.

Theorem 5.2 *Let A be a G-sequence. Then*

$$m(G) = min\{m(G(W)) \,|\, W \in \mathcal{W}_m(A)\}.$$

Proof. Let Y be a G-sequence with $m(G(Y)) = m(G)$. If $m(G) \neq min\{m(G(W)) \,|\, W \in \mathcal{W}_m(A)\}$ then $Y \notin \mathcal{W}_m(A)$, *i.e.*, $Y\!\uparrow > m(G(A))$. So $m(G(Y)) \geq Y\!\uparrow > m(G(A))$, *i.e.*, $m(G) > m(G(A))$, a contradiction to the minimality of $m(G)$. Hence the result. ∎

Using Theorem 5.2 we have:
Algorithm $\mathfrak{m}(G)$: Compute $\mathfrak{m}(G)$

(1) `Let` $A_0 = (a_1, a_2, \ldots, a_n)$ `be a` G`-sequence.`

(2) `Compute` $m(G(A_0))$.

(3) `List` $\mathcal{W}_m(A_0)$ `in lexicographic order.`

(4) `For each` $W \in \mathcal{W}_m(A_0)$ `compute` $m(G(W))$:

`IF we find` $W = A_1$ `with` $m(G(A_1)) < m(G(A_0))$
`THEN let` $A_0 = A_1$ `at Step (2) and repeat`
`ELSE output` $\mathfrak{m}(G) = m(G) = m(G(A_0))$.

Definition: m-minimum G-sequence; witness for $\mathfrak{m}(G)$
Let $Y = (y_1, y_2, \ldots, y_n)$ be a G-sequence. Then Y is a m-minimum G-sequence if $m(G(Y)) = \mathfrak{m}(G)$; we also call Y a witness for $\mathfrak{m}(G)$.

Once $\mathfrak{m}(G)$ is known we can find *all* m-minimum G-sequences, *i.e.*, all witnesses for $\mathfrak{m}(G)$.

Theorem 5.3 *Let Y be a m-minimum G-sequence. Then all G-sequences with $m(G(W)) = \mathfrak{m}(G)$ lie in $\mathcal{W}_m(Y)$.*

Proof. Let W be a m-minimum G-sequence, then $m(G(W)) = \mathfrak{m}(G)$. So $W\!\uparrow \leq m(G(W)) = \mathfrak{m}(G) = m(G(Y))$, *i.e.*, $W \in \mathcal{W}_m(Y)$. ∎

The proof of the following Theorem is similar to the proof of Theorem 4.5.

Theorem 5.4

(i) Let μ be a m-minimum EMI of G. Then $1 \in \mathbb{N}$ appears as a label of G.

(ii) Let $Y = (y_1, y_2, \ldots, y_n)$ be a m-minimum G-sequence. Then either $y_i = 1$ for some $y_i \in Y$ or $\mathcal{P}(Y)\!\uparrow +1 = k(G(Y))$, (not both). ∎

Remark: See Table 3, $G = K_2$, where both cases of Theorem 5.4(ii) are illustrated. Theorem 5.4(ii) gives us a quick method to compute $k(G(Y))$ for a m-minimum G-sequence Y when $1 \notin Y$.

Definition: $witness(\mathfrak{m}(G))$

$$witness(\mathfrak{m}(G)) = \{W \,|\, W \textit{ is a } m\!-\!\textit{minimum } G\!-\!\textit{sequence}\}.$$

We use Theorem 5.4(ii) in the following algorithm where we assume that $\mathfrak{m}(G)$ is known.

Algorithm witness($\mathfrak{m}(G)$): Find $witness(\mathfrak{m}(G))$

(1) `Let` $Y = (y_1, y_2, \ldots, y_n)$ `be a` m`-minimum` G`-sequence.`

(2) `List all` $W = (w_1, w_2, \ldots, w_n) \in \mathcal{W}_m(Y)$ `with some` $w_i = 1$ `or` $\mathcal{P}(Y)\!\uparrow +1 = k(G(W))$`, (not both).`

(3) `For these` W `compute` $m(G(W))$`.`

`IF` $m(G(W)) = \mathfrak{m}(G)$

`THEN output` W `into` $witness(\mathfrak{m}(G))$

`ELSE reject` W`.`

6 Results for G with $n = 2, 3, \ldots, 7$, double-witnesses

In this section, for a fixed graph G with $n = 2, 3, \ldots, 7$ vertices we present our results from Algorithm $\kappa(G)$ and Algorithm $\mathfrak{m}(G)$. (For a typical fixed G, the witnesses from Algorithm $witness(\kappa(G))$ and Algorithm $witness(\mathfrak{m}(G))$ are too numerous to list.)

All the 1252 simple graphs G with up to 7 vertices are listed and numbered as $G1, G2, \cdots, G1252$ in Read and Wilson [10]. We use this numbering system, and for graph $G\#$ we compute the quadruple: $(G\#, \kappa(G\#), \mathfrak{m}(G\#), def(G\#))$.

Definitions: (k, m)-minimum G-sequence; double-witness, (k, m)-minimum EMI of G; double-witness, $double\!-\!witness(G)$

(1) Let $A = (a_1, a_2, \ldots, a_n)$ be a G-sequence. Then A is a (k, m)-minimum G-sequence if both $k(G(A)) = \kappa(G)$ and $m(G(A)) = \mathfrak{m}(G)$; we also call A a double-witness.

(2) Let μ be an EMI of G. Then μ is a (k, m)-minimum EMI of G if both $k_\mu = \kappa(G)$ and $m_\mu = \mathfrak{m}(G)$; we also call μ a double-witness.

(3) $double\!-\!witness(G) = \{W \mid W \; is \; a \; (k, m)\!-\!minimum \; G\!-\!sequence\}$.

Algorithm double-witness(G)**:** Find $double\!-\!witness(G)$

`For each` $X \in witness(\kappa(G))$ `compute` $m(G(X))$.

`IF` $m(G(X)) = \mathfrak{m}(G)$

`THEN output` X `into` $double\!-\!witness(G)$

`ELSE reject` X.

Remark: If G-sequence A is a double-witness we use **bold** numbers in A.

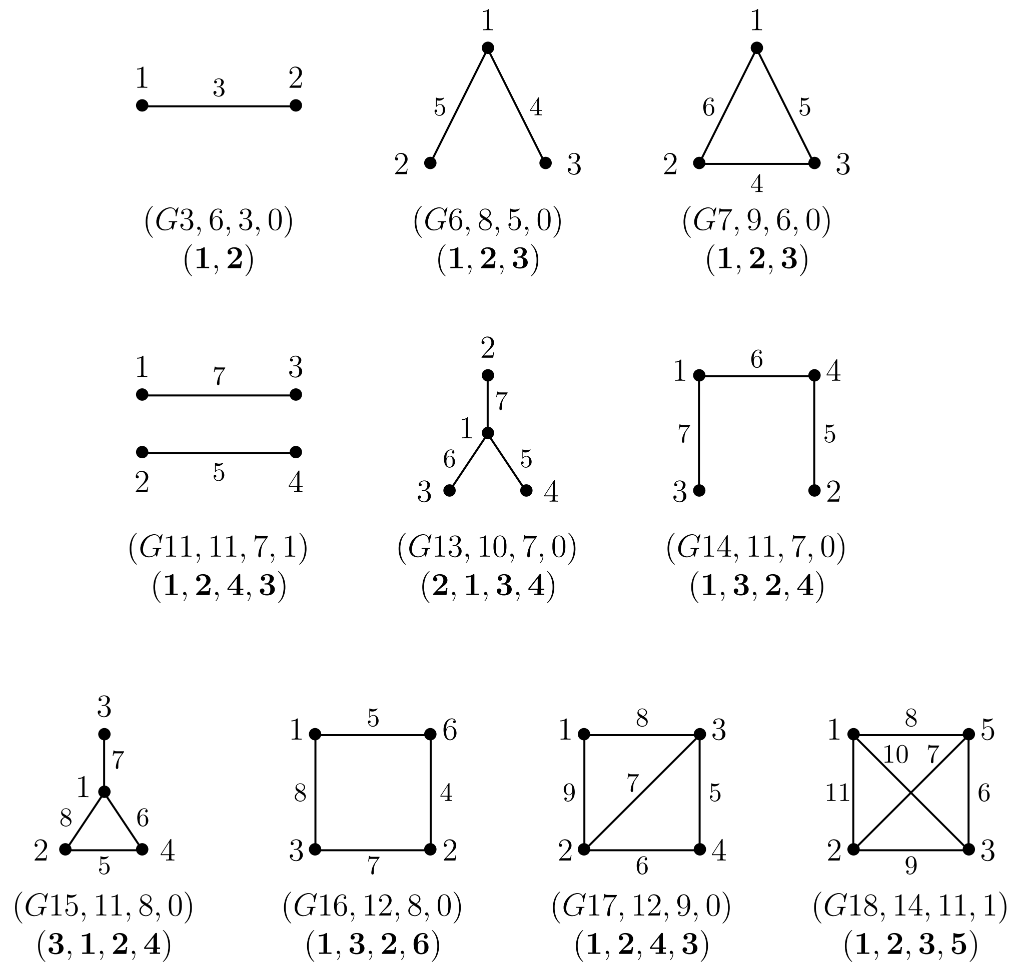


Figure 2: All graphs G with $n = 2, 3$, or 4 vertices. Each G is labelled with a (k, m)-minimum EMI. Below G is its $(G, \kappa(G), \mathfrak{m}(G), def(G))$-quadruple, and the (k, m)-minimum G-sequence (double-witness) consisting of the vertex labels. The smallest graph G without an EMTL ($def(G) > 0$) is $G11 = 2K_2$, the second smallest is $G18 = K_4$.

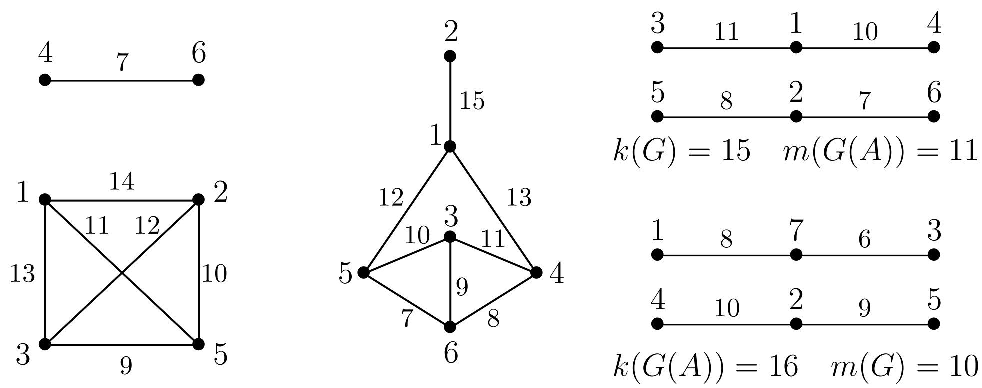
(G26, 13, 8, 0)	(G29, 12, 9, 0)	(G30, 13, 9, 0)	(G31, 13, 9, 0)
(G32, 14, 10, 1)	(G34, 13, 10, 0)	(G35, 13, 10, 0)	(G36, 14, 10, 0)
(G37, 14, 10, 0)	(G38, 14, 10, 0)	(G40, 14, 11, 0)	(G41, 14, 11, 0)
(G42, 14, 11, 0)	(G43, 14, 11, 0)	(G44, 15, 11, 0)	(G45, 15, 12, 0)
(G46, 15, 12, 0)	(G47, 15, 12, 0)	(G48, 15, 12, 0)	(G49, 16, 13, 0)
(G50, 16, 13, 0)	(G51, 17, 14, 0)	(G52, 18, 15, 0)	

Table 1. The $(G, \kappa(G), \mathfrak{m}(G), def(G))$-quadruple for the 23 graphs G on $n = 5$ vertices.

(G61, 15, 9, 0)	(G68, 15, 11, 1)	(G69, 15, 10, 0)	(G70, 15, 10, 0)
(G77, 14, 11, 0)	(G78, 15, 11, 0)	(G79, 15, 11, 0)	(G80, 15, 11, 0)
(G81, 15, 11, 0)	(G82, 15, 11, 0)	(G83, 16, 11, 0)	(G84, 16, 12, 1)
(G85, 16, 11, 0)	(G92, 15, 12, 0)	(G93, 15, 12, 0)	(G94, 15, 12, 0)
(G95, 15, 12, 0)	(G96, 16, 12, 0)	(G97, 16, 12, 0)	(G98, 16, 12, 0)
(G99, 16, 12, 0)	(G100, 16, 12, 0)	(G101, 16, 12, 0)	(G102, 16, 12, 0)
(G103, 16, 12, 0)	(G104, 16, 12, 0)	(G105, 17, 12, 0)	(G106, 18, 13, 1)
(G111, 16, 13, 0)	(G112, 16, 13, 0)	(G113, 16, 13, 0)	(G114, 16, 13, 0)
(G115, 17, 13, 0)	(G116, 17, 14, 1)*	(G117, 16, 13, 0)	(G118, 16, 13, 0)
(G119, 16, 13, 0)	(G120, 16, 13, 0)	(G121, 17, 13, 0)	(G122, 16, 13, 0)
(G123, 17, 13, 0)	(G124, 16, 13, 0)	(G125, 17, 13, 0)	(G126, 17, 13, 0)
(G127, 17, 13, 0)	(G128, 17, 13, 0)	(G129, 17, 13, 0)	(G130, 18, 13, 0)
(G133, 18, 15, 1)	(G134, 17, 14, 0)	(G135, 17, 14, 0)	(G136, 17, 14, 0)
(G137, 17, 14, 0)	(G138, 17, 14, 0)	(G139, 17, 14, 0)	(G140, 17, 14, 0)
(G141, 17, 14, 0)	(G142, 17, 14, 0)	(G143, 18, 15, 1)*	(G144, 17, 14, 0)
(G145, 17, 14, 0)	(G146, 18, 14, 0)	(G147, 17, 14, 0)	(G148, 17, 14, 0)
(G149, 17, 14, 0)	(G150, 18, 14, 0)	(G151, 17, 14, 0)	(G152, 18, 14, 0)
(G153, 18, 14, 0)	(G154, 18, 14, 0)	(G156, 18, 15, 0)	(G157, 18, 15, 0)
(G158, 18, 15, 0)	(G159, 18, 15, 0)	(G160, 18, 15, 0)	(G161, 18, 15, 0)
(G162, 18, 15, 0)	(G163, 18, 15, 0)	(G164, 18, 15, 0)	(G165, 18, 15, 0)
(G166, 18, 15, 0)	(G167, 18, 15, 0)	(G168, 18, 15, 0)	(G169, 18, 15, 0)
(G170, 18, 15, 0)	(G171, 18, 15, 0)	(G172, 18, 15, 0)	(G173, 18, 15, 0)
(G174, 18, 15, 0)	(G175, 19, 15, 0)	(G177, 19, 16, 0)	(G178, 19, 16, 0)
(G179, 19, 16, 0)	(G180, 19, 16, 0)	(G181, 19, 16, 0)	(G182, 19, 16, 0)
(G183, 19, 16, 0)	(G184, 19, 16, 0)	(G185, 19, 16, 0)	(G186, 19, 16, 0)
(G187, 19, 16, 0)	(G188, 19, 16, 0)	(G189, 19, 16, 0)	(G190, 20, 17, 1)
(G191, 20, 17, 0)	(G192, 20, 17, 0)	(G193, 20, 17, 0)	(G194, 20, 17, 0)
(G195, 21, 17, 0)	(G196, 20, 17, 0)	(G197, 20, 17, 0)	(G198, 20, 17, 0)
(G199, 20, 17, 0)	(G200, 21, 18, 0)	(G201, 22, 19, 1)	(G202, 21, 18, 0)
(G203, 21, 18, 0)	(G204, 22, 19, 1)	(G205, 22, 19, 0)	(G206, 22, 19, 0)
(G207, 23, 20, 0)	(G208, 25, 21, 0)		

Table 2. The $(G, \kappa(G), \mathfrak{m}(G), def(G))$-quadruple for the 122 graphs G on $n = 6$ vertices.

Remark: In Table 2 the two graphs $G116$ and $G143$ marked with a * each have $def(G) = 1$; thus we have found two new graphs G on 6 vertices with $def(G) = 1$. See the incomplete list of 7 graphs G on 6 vertices with $def(G) = 1$ in [5]; the complete list is $\{G68, G84, G106, G116, G133, G143, G190, G201, G204\}$. Graph $G116$ is $K_2 \cup K_4$: we have also confirmed that $G116$ does not have an EMTL ($def(G116) > 0$) by exhaustive search without the aid of a computer. Graph $G143$ has 8 edges and odd degrees $[1, 3, 3, 3, 3, 3]$ so we may also use Theorem 1 of Ringel and Llado [11] to confirm that $G143$ does not have an EMTL.



$(G116, 17, 14, 1)$ $(G143, 18, 15, 1)$ $(G70, 15, 10, 0)$

Figure 3: Two new graphs G on 6 vertices each with $def(G) = 1$; $G = 2P_3$, the smallest graph without a double witness, first a k-minimum EMI and then a m-minimum EMI are shown. (The m-minimum EMI is an EMTL.)

Remark: A graph may have a large number of double-witnesses, *eg.*, graph $G77$ on 6 vertices has 3840 double-witnesses. The smallest graph without a double-witness is $G70 = 2P_3$, and the next smallest is $G79 = B_{2,2}$. Both these graphs G have an EMTL and satisfy $15 = \kappa(G) < \kappa_{EMTL}(G) = 16$, see Figs. 1 and 3. The smallest graph without both a double-witness and an EMTL is $G106 = 2K_3$.

Theorem 6.1 *Let G have an EMTL. Then G has a double-witness if and only if $\kappa(G) = \kappa_{EMTL}(G)$.*

Proof. Graph G has an EMTL so $\mathfrak{m}(G) = n + e$.

For the forward implication: Let Z be a double-witness for G. So $m(G(Z)) = \mathfrak{m}(G) = n + e$, *i.e.*, $G(Z)$ is an EMTL. Thus $\kappa_{EMTL}(G) \leq k(G(Z)) = \kappa(G)$. Hence, from Equation (1), $\kappa(G) = \kappa_{EMTL}(G)$.

For the backward implication: Let X be a witness for $\kappa_{EMTL}(G)$, then $k(G(X)) = \kappa_{EMTL}(G) = \kappa(G)$. Also, since $G(X)$ is an EMTL, $m(G(X)) = n + e = \mathfrak{m}(G)$. That is, X is a double-witness for G. ∎

We have a countable class of graphs without a double-witness:

Corollary 6.2 *For even $n \geq 4$ the bi-star $B_{\frac{n-2}{2},\frac{n-2}{2}}$ does not have a double-witness.*

Proof. From the Remark after Theorem 2.4 we have $\kappa(B_{\frac{n-2}{2},\frac{n-2}{2}}) < \kappa_{EMTL}(B_{\frac{n-2}{2},\frac{n-2}{2}})$, the contrapositive of Theorem 6.1 then gives the result. ∎

Note: A file containing the $(G, \kappa(G), \mathfrak{m}(G), def(G))$-quadruples for the 888 graphs G on $n = 7$ vertices is available from the authors. As is a file containing the $(T, \kappa(T), \mathfrak{m}(T), def(T))$-quadruples for the 200 trees T on up to $n = 10$ vertices.

7 $G = K_n$, Well Spread-sequences

When $G = K_n$ a G-sequence is a *Well Spread*-sequence, a WS-sequence; see Kotzig [4], and [6].

An upperbound for both $\kappa(K_n)$ and $\mathfrak{m}(K_n)$ due to Wood [15] is:

$$\kappa(K_n),\ \mathfrak{m}(K_n) \leq (3 + o(1))n^2,$$

and an upperbound for $\kappa(K_n)$ due to Pikhurko [9] is:

$$\kappa(K_n) \leq (2.38... + o(1))n^2.$$

Lower bounds for $\kappa(K_n)$ and $\mathfrak{m}(K_n)$ are given at the end of this section in Theorem 7.6.

We first complete our discussion of graphs G from Theorem 1.1 by dealing with line 1:

Theorem 7.1 *We have $\kappa(K_n) = \kappa_{EMTL}(K_n)$ for $n \in \{2, 3, 5, 6\}$.*

Proof. For each $n \in \{2, 3, 5, 6\}$ compare the value of $\kappa(K_n)$ in Section 6 to the value of $\kappa_{EMTL}(K_n)$ in Theorem 1.1. ∎

Definition: dual of A
Let $A = (a_1, a_2, \ldots, a_n)$ be a WS-sequence. The dual of A is the WS-sequence $A' = (m(K_n(A)) + 1 - a_n, m(K_n(A)) + 1 - a_{n-1}, \ldots, m(K_n(A)) + 1 - a_1)$.

n	$\kappa(K_n)$	$W \in$ witness$(\kappa(K_n))$	$\mathfrak{m}(K_n)$	$W \in$ witness$(\mathfrak{m}(K_n))$, W'	$def(K_n)$
1	1		1		0
2	6	$(\mathbf{1},\mathbf{2})$ $(\mathbf{1},\mathbf{3})$ $(\mathbf{2},\mathbf{3})$	3^*	$(\mathbf{1},\mathbf{2})^*$, $(\mathbf{2},\mathbf{3})^*$ $(\mathbf{1},\mathbf{3})^*$, $(\mathbf{1},\mathbf{3})^*$	0
3	9	$(\mathbf{1},\mathbf{2},\mathbf{3})$	6^*	$(\mathbf{1},\mathbf{2},\mathbf{3})^*$, $(4,5,6)^*$ $(1,3,5)^*$, $(2,4,6)^*$	0
4	14	$(\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{5})$ $(\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{7})$ $(\mathbf{1},\mathbf{2},\mathbf{4},\mathbf{7})$	11^*	$(\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{5})$, $(7,9,10,11)$ $(\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{7})$, $(5,9,10,11)$ $(\mathbf{1},\mathbf{2},\mathbf{4},\mathbf{7})$, $(5,8,10,11)$ $(1,3,4,5)$, $(7,8,9,11)$ $(1,3,5,8)$, $(4,7,9,11)$ $(1,4,5,9)$, $(3,7,8,11)$ $(1,6,7,8)$, $(4,5,6,11)$ $(1,6,7,9)$, $(3,5,6,11)$ $(2,3,6,9)$, $(3,6,9,10)$ $(2,4,6,10)$, $(2,6,8,10)$	1
5	18	$(\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{5},\mathbf{9})$	15^*	$(\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{5},\mathbf{9})^*$, $(7,11,13,14,15)^*$ $(1,8,9,10,12)^{**}$, $(4,6,7,8,15)^{**}$	0
6	25	$(\mathbf{1},\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{9},\mathbf{14})$ $(1,2,3,5,8,13)$	21^*	$(\mathbf{1},\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{9},\mathbf{14})^*$, $(8,13,17,18,19,21)^*$ $(2,6,7,8,10,18)^{**}$, $(4,12,14,15,16,20)^{**}$	0

Table 3. Values of $\kappa(K_n)$, $\mathfrak{m}(K_n)$, and $def(K_n)$ for $1 \le n \le 6$, and all witnesses.

Remark: In Table 3 the values and sequences marked * first appeared in Kotzig and Rosa [5] and [6], and the sequences marked ** first appeared in Section 7.1 of [14], see also Section 2.3.3 of [13]. All remaining values and sequences are new. (The sequence $(8, 13, 17, 18, 19, 21)$ was printed erroneously with the '13' as '11' in both [14] and [13].) As before, double-witnesses appear in **bold**. Note that the m-minimum WS-sequences appear in W, W' pairs.

n	$\kappa(K_n)$	$W \in$ witness$(\kappa(K_n))$	$\mathfrak{m}(K_n)$	$W \in$ witness$(\mathfrak{m}(K_n))$, W'	$def(K_n)$
7	38	$(1,2,3,5,8,13,21)$ $(1,4,5,7,9,14,21)$	32	$(3,4,5,7,10,15,23)$, $(10,18,23,26,28,29,30)$	4
8	51	$(\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{11},\mathbf{16},\mathbf{22},\mathbf{28})$	46	$(\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{11},\mathbf{16},\mathbf{22},\mathbf{28})$, $(19,25,31,36,41,43,44,45)$ $(3,5,6,7,11,16,23,30)$, $(17,24,31,36,40,41,42,44)$	10
9	71	$(1,2,3,5,9,15,20,29,38)$ $(2,3,4,6,12,17,22,29,41)$	64	$(1,10,11,12,14,19,27,33,39)$, $(26,32,38,46,51,53,54,55,64)$ $(2,10,12,13,14,18,27,34,41)$, $(24,31,38,47,51,52,53,55,63)$ $(3,5,6,7,11,16,25,32,39)$, $(26,33,40,49,54,58,59,60,62)$ $(10,12,13,14,18,23,32,39,46)$, $(19,26,33,42,47,51,52,53,55)$ $(12,13,14,16,20,26,31,36,52)$, $(13,29,34,39,45,49,51,52,53)$	19
10	89	$(\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{5},\mathbf{9},\mathbf{16},\mathbf{25},\mathbf{30},\mathbf{35},\mathbf{47})$	86	$(\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{5},\mathbf{9},\mathbf{16},\mathbf{25},\mathbf{30},\mathbf{35},\mathbf{47})$, $(40,52,57,62,71,78,82,84,85,86)$ $(2,8,9,10,14,18,28,31,42,53)$, $(34,45,56,59,69,73,77,78,79,85)$ $(6,10,14,15,22,24,35,38,41,60)$, $(27,46,49,52,63,65,72,73,77,81)$ $(13,14,15,20,26,29,39,43,47,65)$, $(22,40,44,48,58,61,67,72,73,74)$	31
11	116	(2,3,4,11,26,29,39,43,49,55,60)	$110^{\dagger}$	$(6,7,8,10,14,21,30,35,40,52,70)^{\dagger\dagger}$, $(41,59,71,76,81,90,97,101,103,104,105)$	44
12	140–154		137–150		59–72

Table 4. Values of $\kappa(K_n)$, $\mathfrak{m}(K_n)$, and $def(K_n)$ for $7 \le n \le 11$, and all witnesses, except for the value $\mathfrak{m}(K_{11}) = 110$ where the sequence marked †† is unlikely to be the only witness; a large portion of the search space was left unsearched. For $n = 12$ we give lower and upper bounds; see below.

Remark: In Table 4 for $\mathfrak{m}(K_{11}) = 110^{\dagger}$ see the comments involving $\rho^*(n)$ below. We have answered Research Problem 2.2 in Section 2.3.4 of [13] which asks to find $\mathfrak{m}(K_7)$ and $\mathfrak{m}(K_8)$.

If $A = (a_1, a_2, \ldots, a_n)$ is a WS-sequence then without loss of generality we let $1 \le a_1 < a_2 < \cdots < a_n$. Thus, see Definitions (4) and (7) of Section 3, we have:

$$k(K_n(A)) \ge a_{n-1}+a_n+1 \text{ and } m(K_n(A)) = max\{a_n, k(K_n(A))-a_1-a_2\}. \quad (3)$$

Definitions: $\rho(A)$, $\rho^*(n)$ (See [4] and [6].)

(1) Let $A = (a_1, a_2, \ldots, a_n)$ be a WS-sequence. Then $\rho(A) = a_n + a_{n-1} - a_2 - a_1 + 1$ is the span of pairwise sums of A.

(2) $\rho^*(n) = min\{\rho(A) \,|\, A$ is a WS-sequence of length $n\}$, is the smallest $\rho(A)$ amongst all WS-sequences A of length n.

Theorem 7.2 *Let $A = (a_1, a_2, \ldots, a_n)$ be a WS-sequence. Then*

(i) $m(K_n(A)) \geq \rho(A)$,

(ii) $\mathfrak{m}(K_n) \geq \rho^*(n)$.

Proof. (i) Using Equation (3) twice gives $m(K_n(A)) \geq k(K_n(A)) - a_1 - a_2 \geq a_{n-1} + a_n + 1 - a_1 - a_2 = \rho(A)$. For (ii) let Y be a WS-sequence with $m(K_n(Y)) = \mathfrak{m}(K_n)$. Then $\mathfrak{m}(K_n) = m(K_n(Y)) \geq \rho(Y) \geq \rho^*(n)$. ∎

Remark: For $n = 2, 3, \ldots, 8$ the values of $\rho^*(n)$ were first computed by hand in [4]; these values were verified by computer and extended to $n = 12$ in [8], see also Section 2.3.4 of [13]. In particular, $\rho^*(11) = 110$. As mentioned in the caption for Table 4 and the Remark following Table 4, for $n = 11$ the WS-sequence $A = (6, 7, 8, 10, 14, 21, 30, 35, 40, 52, 70)^{\dagger\dagger}$ has $m(K_{11}(A)) = 110$, thus $\mathfrak{m}(K_{11}) \leq 110$. But from Theorem 7.2(ii) we have $\mathfrak{m}(K_{11}) \geq \rho^*(11) = 110$. Thus $\mathfrak{m}(K_{11}) = 110$ as given in Table 4. The inequality of Theorem 7.2(ii) is tight for $n = 11$.

Definitions: $\sigma(A)$, $\sigma^*(n)$ (See [4] and [6].)

(1) Let $A = (a_1, a_2, \ldots, a_n)$ be a WS-sequence. Then $\sigma(A) = a_n - a_1 + 1$ is the span of A.

(2) $\sigma^*(n) = min\{\sigma(A) \,|\, A \text{ is a WS-sequence of length } n\}$, is the smallest $\sigma(A)$ amongst all WS-sequences A of length n.

Remark: As for $\rho^*(n)$, for $n = 2, 3, \ldots, 8$ values of $\sigma^*(n)$ first appeared in [4]; and were extended to $n = 12$ in [8].

Lemma 7.3 *Let $A = (a_1, a_2, \ldots, a_n)$ be a WS-sequence. Then $a_i \geq \sigma^*(i)$ for each $i = 1, 2, \ldots, n$.*

Proof. For each $i = 1, 2, \ldots, n$ consider the WS-sequence $A_i = (a_1, a_2, \ldots, a_i)$. We have $\sigma(A_i) = a_i - a_1 + 1 \geq \sigma^*(i)$. So $a_i \geq \sigma^*(i) + a_1 - 1$. But $a_1 \geq 1$, and hence the result. ∎

Theorem 7.4 *We have* $\kappa(K_n) \geq \sigma^*(n-1) + \sigma^*(n) + 1$.

Proof. Let $X = (x_1, x_2, \ldots, x_n)$ be a WS-sequence with $k(K_n(X)) = \kappa(K_n)$. Using Equation (3) then Lemma 7.3, we have $\kappa(K_n) = k(K_n(X)) \geq x_{n-1} + x_n + 1 \geq \sigma^*(n-1) + \sigma^*(n) + 1$. ∎

Theorem 7.5 *We have* $\kappa(K_n) \geq \mathfrak{m}(K_n) + 3$.

Proof. Let μ be a k-minimum EMI of K_n, *i.e.*, $k_\mu = \kappa(K_n)$, with largest label $m_\mu \geq \mathfrak{m}(K_n)$. Whether label m_μ is on a vertex or edge, the magic sum k_μ is the sum of this label and two other labels. Hence $\kappa(K_n) = k_\mu \geq m_\mu + 1 + 2 \geq \mathfrak{m}(K_n) + 3$. ∎

Note: From Theorem 7.2(ii) and [8] we have $\mathfrak{m}(K_{12}) \geq \rho^*(12) = 137$. And then from Theorem 7.5 we have $\kappa(K_{12}) \geq \mathfrak{m}(K_{12}) + 3 \geq 140$. The WS-sequence $S = (1, 3, 5, 6, 9, 21, 32, 41, 51, 58, 65, 79)$ has $k(K_{12}(S)) = 154$ and $m(K_{12}(S)) = 150$. Thus $\kappa(K_{12}) \leq 154$ and $\mathfrak{m}(K_{12}) \leq 150$.

Hence $140 \leq \kappa(K_{12}) \leq 154$ and $137 \leq \mathfrak{m}(K_{12}) \leq 150$ as given in Table 4.

We finish this section with lower bounds for $\kappa(K_n)$ and $\mathfrak{m}(K_n)$:

Theorem 7.6 *For* $n \geq 13$, *we have*

(i) $\kappa(K_n) \geq n^2 - 4n + 13$,

(ii) $\mathfrak{m}(K_n) \geq n^2 - 5n + 14$.

Proof. (i) From [4] we have $\sigma^*(n) \geq 4 + \binom{n-1}{2}$; Theorem 7.4 then gives the result.
(ii) Also from [4] we have $\rho^*(n) \geq n^2 - 5n + 14$; Theorem 7.2(ii) then gives the result. ∎

8 Miscellaneous

Comment on algorithms: The first four algorithms presented in this paper each include an operation which lists the G-sequences in the sets $\mathcal{W}_k(A)$, or $\mathcal{W}_m(A)$, in lexicographical order. Then each G-sequence is evaluated to determine if it produces an equal, or smaller, value as the presently smallest known value, for $k(G(A))$, or $m(G(A))$, or if it is a witness for $\kappa(G)$, or for $\mathfrak{m}(G)$. The actual software implementations of these algorithms do not follow these instructions exactly as it proved much easier to simply generate (and evaluate) all reasonable G-sequences while maintaining which ones are currently the best, and then reporting the final results once the entire set of reasonable G-sequences has been exhausted. This brute force enumeration strategy is one whose efficiency is greatly improved by the known bounds for $\sigma^*(i)$, which denotes the smallest that the i-th term of any WS-sequence of length at least i can be, see Lemma 7.3.

Three new integer sequences: Tables 3 and 4 provide us with three integer sequences that do not appear in the On-Line Encyclopedia of Integer Sequences, Sloane [12]. We have sent the first 11 terms of each sequence to [12]. They are: $\{\kappa(K_n) \,|\, n \geq 1\} = \{1, 6, 9, 14, 18, 25, 38, 51, 71, 89, 116, \ldots\}$, $\{\mathfrak{m}(K_n) \,|\, n \geq 1\} = \{1, 3, 6, 11, 15, 21, 32, 46, 64, 86, 110, \ldots\}$, and $\{def(K_n) \,|\, n \geq 1\} = \{0, 0, 0, 1, 0, 0, 4, 10, 19, 31, 44, \ldots\}$.

Further Research and Questions:

(1) Determine the exact values of $\kappa(K_{12})$ and $\mathfrak{m}(K_{12})$; and extend Table 4 for $n \geq 13$.

(2) For every even $n \geq 2$ does K_n have a double-witness? It does for $n = 2, 4, 6, 8$, and 10.

(3) For $n = 2, 3, 4, 5$, and 6 the following is true: $\kappa(G) < \kappa(K_n)$ and $\mathfrak{m}(G) < \mathfrak{m}(K_n)$, for all $G \neq K_n$. Is it true for all n?

(4) Find $\kappa(G)$ and $\mathfrak{m}(G)$ for other graphs G, and for graphs $G_n \in \mathcal{G}$ for a class of graphs $\mathcal{G}$. In particular, for classes $\mathcal{G}$ whose members have an EMTL; *eg.*, $\mathcal{G} = \{K_{\frac{n}{2},\frac{n}{2}} \mid n \text{ even and } \geq 2\}$, and $\mathcal{G} = \{K_{m,n} \,|\, m, n \geq 1\}$. See [2] for many other such classes.

(5) For $G_n \in \mathcal{G}$ investigate the quantity $\kappa_{EMTL}(G_n) - \kappa(G_n)$. Can $\kappa_{EMTL}(G_n) - \kappa(G_n) > cn^2$, for some constant $c > 0$ and for sufficiently large n? See the Remark after Theorem 2.4.

(6) Improve the lower bounds on $\kappa(K_n)$ and $\mathfrak{m}(K_n)$, see Theorem 7.6.

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