

Harmonic evolutions on graphs

Jerzy Kocik

Department of Mathematics

Southern Illinois University

Carbondale, IL 62901, USA

jkocik@siu.edu

Abstract

We define the harmonic evolution of states of a graph by iterative application of the harmonic operator (Laplacian over Z_2). This provides graphs with a new geometric context and leads to a new tool to analyze them. The digraphs of evolutions are analyzed and classified. This construction can also be viewed as a certain topological generalization of cellular automata.

MSC: 05C50, 15A33, 05C75, 05C85.

1 Introduction

The Laplacian of a graph is a well known tool to study graphs. Typically, we retrieve its eigenvalues and deduce certain graph's properties [1, 2, 5]. In this paper we rather analyze the graph's "harmonic operator", the Laplacian modulo Z_2 , or — more concretely — the monoid it generates. Interestingly, the emerging algebraic properties may be interpreted pictorially and visualized. We define a cellular automaton on an arbitrary (simple) graph as follows.

The Harmonic Game ("game of light"). Choose a graph to represent your "**universe**". Each vertex may be in one of the two states: "on" or "off" (excited or not excited). Any initial state of a graph will evolve stepwise according to certain "**laws**":

1. If a non-excited cell has an odd number of excited neighbors, it becomes excited.
2. If an excited cell has an odd number of non-excited neighbors, it will remain excited, otherwise it relaxes and goes to rest.

Examples of evolutions on various graphs may be inspected in Figures 1 and 2.

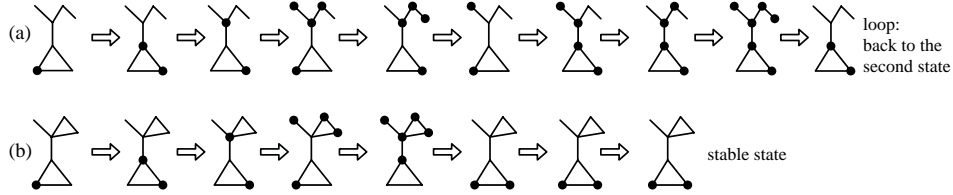


Figure 1: Examples of harmonic evolution on graphs (“game of light”).

Clearly, each process has to eventually end up in a loop, consisting possibly of a single element, like Fig. 1b. Even small change in structure may radically change the size of the loop and of the path towards the loop (cf. Fig. 1a and 1b).

The question is — what are all possible evolutions on a given graph G due to different initial states. Laborious exercise would recover a digraph whose nodes are the states (2^n of them, n =number of vertices in G), and whose directed edges indicate the consecutive states. See Figure 3 for an example. It will be called the **evolution digraph** of G and denoted G^* . It turns out that evolution

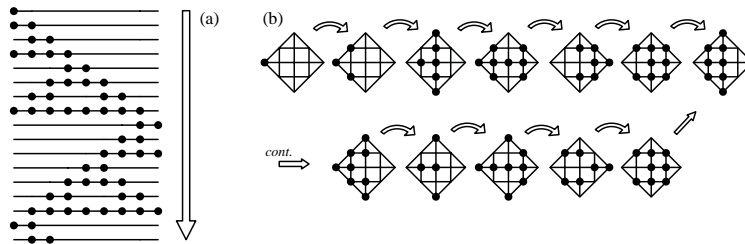


Figure 2: Game is like a propagation (a) a string with 9 nodes (b) a diamond shaped graph.

digraphs have the same general structure: each G^* consists of a certain number of closed loops, and to every node in a loop a certain tree is attached, constituted by states descending towards it. Interestingly, every tree in G^* has the same form. Other examples of graphs and their evolution digraphs are shown in Figure 5.

The structure of these trees and of the loops is concealed in the harmonic operator and its powers. We will describe an algorithm that uncovers it.

The harmonic game reminds one of Conway’s “game of life.” But things are here more fundamental. Unlike the usual cellular automata, the rules are not arbitrary, but are derived from the topologically invariant notions, that of boundary

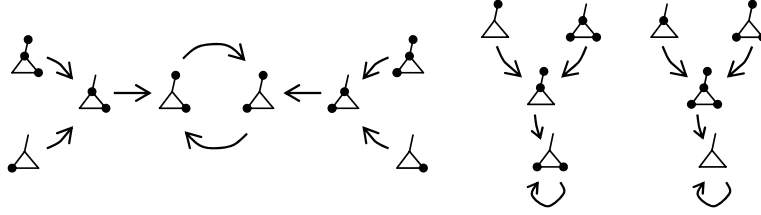


Figure 3: Evolution digraph G^* of a graph G .

and coboundary. They represent the action of the harmonic operator a . Moreover, the rules and the processes in Figures 1 and 2 present only the surface of the actual dynamic, because both the vertices and the edges are actually involved in the process.

Motivations for studying Harmonic games (HG):

1. (Graph theory) A tool to analyze graphs in an alternative way to that of Laplacian's eigenvalues. The behavior of the states convey information on eigenspaces of powers of a . In a sense the harmonic game is a method to illustrate the structure of eigenspaces of the Laplacian by trees, loops, etc.
2. (CA) Although we see it as a graph-theoretic study, we view HG as a step beyond the regularity of grids used in cellular automata. And yet another difference: HG conceals actually two intertwined evolutions, one on vertices and one on edges, as explained in the next section (the rules presented here present only a surface of the dynamics).
3. (Metaphor) A toy model of reality. HG is a metaphor of a typical theory of theoretical physics. (The graph itself is like a *protogeometry*: locations and connections only). One is reminded of Wheeler's search for the "Law without Law" [6] or his "pre-geometry" [7].

Remark: If biologists have their mathematical toy in the "game of life", HG may be viewed as a physicist's toy, a metaphor for "light propagation": The rules for a one-step change defined by the harmonic operator are styled after the harmonic operator in differential geometry (and Maxwell's theory of light). Harmonic evolution has indeed the flavor of light "propagation" (see Fig. 2). It actually obeys "Huygens principle": the sum (mod 2) of two evolutions is an evolution. Note other features: the superposition principle, periodicity induced by boundary conditions, duality in propagation (like in electromagnetic wave), simplicity of laws, their topological foundations, etc.

2 Harmonic Evolution

Let $G = (\text{Ver}(G), \text{Edg}(G))$ be a simple graph (no loops, no multiple edges [5]) with the set of vertices $\text{Ver}(G)$ and the set of edges $\text{Edg}(G)$. We define a *state* (respectively *co-state*) as a subset of vertices (respectively edges) of G . Denote the family of all subsets of $\text{Ver}(G)$ by \mathbb{V}_G , and that of $\text{Edg}(G)$ and by \mathbb{E}_G . Define addition as the symmetric difference $A + B =: (A - B) \cup (B - A)$ for any two subsets A and B (“adding modulo two”).

The *boundary* operator $\partial : \mathbb{V}_G \rightarrow \mathbb{E}_G$ sends maps into co-states, namely for any $S \in \mathbb{V}_G$, we define ∂S as the sum of the sets of edges adjacent to vertices of S , added modulo two. Similarly, a *co-boundary* operator $\delta : \mathbb{E}_G \rightarrow \mathbb{V}_G$ maps co-states into states, namely for $\sigma \in \mathbb{E}_G$ we define $\delta\sigma$ as the set of vertices adjacent to the edges of σ , added modulo two (see Figure 4).

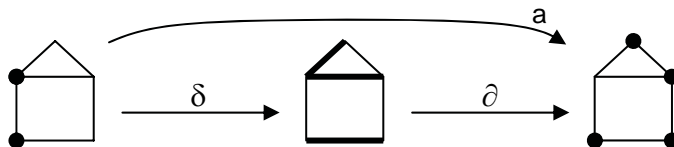


Figure 4: Boundary ∂ , co-boundary δ , and harmonic map $a = \partial\delta$

By analogy to Hodge theory in differential geometry, the *harmonic map* is defined as the composition $a = \partial\delta + \delta\partial$. In particular, if restricted to states, the harmonic operator becomes

$$a = \partial\delta \in \text{End } \mathbb{V}_G \quad (2.1)$$

By iterative application of the harmonic operator to an initial state $S \in \mathbb{V}_G$, we obtain a sequence $\{S, aS, a^2S, \dots\}$ that we call the *evolution* of S , examples of which are in Fig. 1 and 2.

For a given graph G , all evolutions form an *evolution digraph* G^* , the nodes of which are the states, \mathbb{V}_G , and the oriented edges of which are of the form (S, aS) (see Fig. 3). The problem is to determine the evolution digraph of a given graph.

It turns out that digraph G^* consists of certain number of cycles, (later collectively called the *loop ensemble*), to each node of which attached is a copy of a descending tree (later called the *characteristic tree* of G). Digraph G^* is a source of invariants of G .

Before we go on, inspect Figure 3 and 5 that present evolution digraphs for a few small graphs (ignore for now the labels in Figure 5). Notice these features: (i) the sum of nodes in the loops is a power of two; (ii) the size of each loop divides the size of the longest loop; (iii) each descending tree has the same structure;

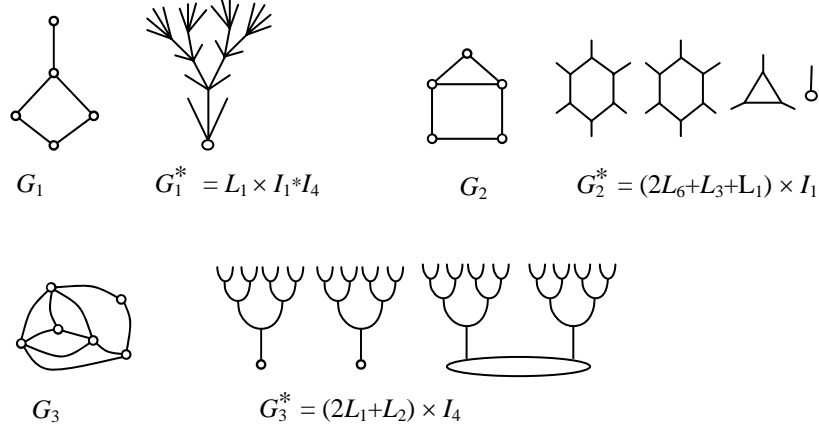


Figure 5: Three graphs and their evolution digraphs.

(iv) the number of nodes in each tree is a power of two; (v) trees have some regularity in shape. Thus the shape of the tree and of the loop part would suffice to reconstruct the digraph, as the arc orientations are unambiguous.

3 Geometry of Graphs

The following ‘geometrization of graphs’ will be exploited. The space of states \mathbb{V}_G of a graph will be viewed as an n -dimensional linear space over \mathbf{Z}_2 , that is, $\mathbb{V}_G \cong \mathbf{Z}_2^n$, where $n = |\text{Ver}(G)|$. The structure of the graph is represented by two \mathbf{Z} -valued scalar products in the space \mathbb{V}_G . One is the natural scalar product g , which for the vectors representing single vertices is $g(v_i, v_j) = \delta_{ij}$. The other is a (possibly degenerated) scalar product A represented in the natural basis by the adjacency matrix, that is $A(v_i, v_j) = 1$ if $(v_i, v_j) \in \text{Ver}(G)$, and $A(v_i, v_j) = 0$ otherwise.

The harmonic operator (2.1) is geometrically an endomorphism of space \mathbb{V}_G , self-adjoint with respect to g . One may easily show that in the natural basis of vertices, it has the matrix form

$$a = A + D ,$$

where D is a diagonal matrix with values 1 at the diagonal entries that correspond to the odd vertices, and 0 otherwise. This is the Laplacian of the graph taken modulo 2. As a matrix, a has an even number of 1’s in every row and every column. Let Ω denote a column of all entries equal to one. Then actually any square matrix with entries 0 and 1 only, satisfying (i) $a^T = a$ and (ii) $a\Omega = 0$, determines a graph.

Evolution of a state $S \in \mathbb{V}$ is given by $S(t) = a^t S$ where $t \in \mathbf{N}$ plays the role of time. Thus, in order to study harmonic evolutions we need to start with the monoid of endomorphisms generated by a . Due to the finiteness of G , the monoid is of the form:

$$M(a) = \{1, a, a^2, \dots, a^k, \dots, a^{k+n-1}\},$$

where $a^{k+n} = a^k$ for some k and n . Part $L(a) =: \{a^k, \dots, a^{k+n-1}\}$ forms a cyclic group, part $T(a) =: \{1, \dots, a^k\}$ is called the *tail* of $M(a)$. We shall use the notation $n = |L(a)|$ and $k = |T(a)|$.

Lemma 3.1 *Operator $\pi = a^p$ with $p = |L(a)| \cdot |T(a)|$ is an orthogonal projection in \mathbb{V} .*

Proof: One may easily show that π is the neutral element of the group $L(a)$. Therefore, $\pi^2 = \pi$, i.e., π is a projection. Orthogonality of π follows from self-adjointness of a . \square

Proposition 3.2 *The space of states \mathbb{V}_G is a direct product of two subspaces*

$$\mathbb{V}_G = T(G) \oplus L(G) \tag{3.1}$$

where $T(G) = \text{Ker } \pi$ and $L(G) = \text{Im } \pi$. Moreover, restriction of a to $T(G)$ is nilpotent, $a^k = 0$, and restriction of a to $L(G)$ is an automorphism of \mathbb{V} .

Lemma 3.1 implies:

Corollary 3.3 *The dimension $\dim L(G)$ is even for any graph G .*

Proof: Indeed, projection π must have two eigenvalues: namely 1 for the subspace $L(G)$, and 0 for $T(G)$. Therefore, a basis of \mathbb{V}_G exists in which π is expressed by a diagonal matrix $\pi_0 = \text{diag}[1, 1, \dots, 0, 0 \dots, 0]$. Trace does not depend on basis; therefore, $\dim L = \text{Tr } \pi_0 = \text{Tr } \pi$. Since $\text{Tr } a^i$ is even for any i , so is $\dim L$. \square

Proposition 3.2 explains the general structure of evolution digraphs. Here is its meaning.

(i) The n -cube $\mathbb{V}_G = \mathbf{Z}_2^n$ may be considered as a digraph $(\mathbb{V}_G, \varphi \rightarrow)$, where the “looped arrow” denotes the relation of succession. That is, for any two vectors (states) $x, y \in \mathbb{V}_G$ we say $x \varphi \rightarrow y$ only if $y = ax$.

(ii) Subspace $T(G) \in \mathbb{V}_G$ is closed under (nilpotent) action of a . It may be, too, considered as a digraph $(T(G), \varphi \rightarrow)$, with a similarly defined relation of succession. Due to nilpotency of a , this digraph has a form of a tree with 0 as the root. We shall call this digraph the **characteristic tree** of G .

(iii) Similarly, the subspace $L(G) \in \mathbb{V}_G$ is closed under action of a , but this time a acts as an automorphism. Thus, as a digraph $(L(G), \varphi)$, it consists of a number of loops (cycles) arising as the orbits of the action of the cyclic group generated by a . We shall call this digraph the **loop ensemble** of G .

(iv) Now, the evolution digraph G^* results as a “semidirect” product of the two,

$$G^* = L(G) \times T(G) = \{L(G) \times T(G), \varphi\}$$

with the succession in $L \times T$ defined

$$(x, y) \varphi (x', y') \text{ if either } (x = x' \text{ and } y \varphi y') \text{ or } (x \varphi x' \text{ and } y = y' = 0).$$

As a digraph, $G^* = T(G) \times L(G)$ will be simply denoted as $G^* = \{T(G), L(G)\}$.

In the following two sections we shall characterize the two digraphs.

4 Classification of Harmonic Trees.

Let $A * B$ denote the digraph obtained as the direct product of two digraphs A and B . That is we set $(a, b) \prec (a', b')$ iff $a \prec b$ in A and $b \prec b'$ in B .

Theorem 4.1 *The characteristic tree of a graph can be uniquely factored into a product of binomial trees:*

$$T(G) = I_1^{b_1} * I_2^{b_2} * \dots * I_k^{b_k} \quad (4.1)$$

where I_i denotes a binomial tree of height i .

Proof: Since a restricted to $T(G)$ is nilpotent, then —by Jordan decomposition theorem— there exists a basis in $T(G)$ such that operator a takes in it a quasi-diagonal form:

$$[a] = \begin{bmatrix} J_1 & & & & \\ & J_2 & & & \\ & & \ddots & & \\ & & & J_i & \\ & & & & \ddots \end{bmatrix} \quad \text{where} \quad J_i = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & & \ddots & \\ & & & & 1 & 0 \end{bmatrix} \quad (4.2)$$

(possibly, some J_i are 1×1 null matrices). This means that the space $T(G)$ decomposes into eigenspaces of the harmonic operator, on which map a acts as a “raising” operator. Namely, denote the vectors that span the subspace E_i corresponding to the sub-matrix J_i by $\{f_1, \dots, f_q\}$. Then a acts :

$$f_i \xrightarrow{a} f_{i+1} \quad \text{and} \quad f_q \xrightarrow{a} 0 \quad (4.3)$$

Note that each f_i (for $i > 1$) is an image of exactly two states, namely f_{i-1} and $f_q + f_{i-1}$. So is, therefore, any sum of these vectors. The state f_1 , and any sum

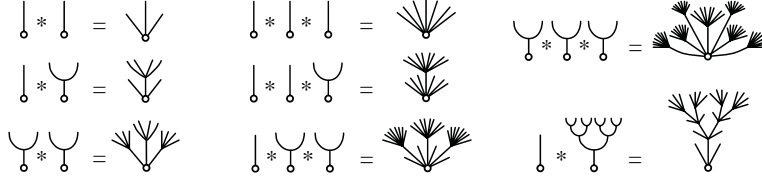


Figure 6: Products of binary trees: a fragment of multiplication table.

$\sum_{i \in I} f_i$ containing f_1 , do not have any anti-images through a . Therefore, the states of the subspace E_i form a binomial tree of height q with 2^q edges, among which half, 2^{q-1} , form the top (starting) nodes. Decomposition of a into linearly independent blocks (4.2) corresponds to multiplication (4.1) of such trees. \square

A few products of binary trees are shown in Fig. 6. Note that the height of the resulting tree is that of the highest tree in the product.

The above theorem implies that the number of the binomial trees in the decomposition (4.1) equals $\dim \text{Ker } a$. Indeed, in the quasi-diagonal form (4.2) of the harmonic matrix, each J_i contributes a single one-dimensional subspace to $\text{Ker } a$. (Clearly $\dim \text{Ker } a|_{T(G)} = \dim \text{Ker } a$, since $\text{Ker } a \subset T(G)$). Another implication is that if $\dim \text{Ker } a = 1$, then $T(G)$ is a binomial tree, $T(G) = I_{|T(a)|}$. If $\dim \text{Ker } a = 2$, then $T(G) = I_{|T(a)|} * I_{\dim T(G) - |T(a)|}$.

The composition of the characteristic tree, that is the exponents of the factorization (4.1), may be deduced by means of the elements of the monoid $M(a)$.

Theorem 4.2 *The multiplicity of the binomial tree I_j in $T(G)$ is given by the following formula*

$$n_j = 2\dim \text{Ker } a^j - \dim \text{Ker } a^{j+1} - \dim \text{Ker } a^{j-1} \quad \text{for } j \leq k \quad (4.4)$$

Proof: The dimensions of kernel of powers of a satisfy the following system of equations:

$$\begin{aligned} \dim \text{Ker } a &= n_1 + n_2 + \dots + n_k \\ \dim \text{Ker } a^2 &= n_1 + 2(n_2 + \dots + n_k) \\ \dim \text{Ker } a^3 &= n_1 + 2n_2 + 3(n_3 + n_4 + \dots + n_k) \\ &\dots \\ \dim \text{Ker } a^k &= n_1 + 2n_2 + 3n_3 + \dots + kn_k \end{aligned} \quad (4.5)$$

Denote $K_i = \dim \text{Ker } a^i$. Then (4.5) is equivalent to a matrix equation $\mathbf{K} = M\mathbf{n}$, where $M_{ij} = \min(i, j)$. The inverse matrix M^{-1} , that is easy to find (here for $k=5$ for illustration):

$$K = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \quad \implies \quad K^{-1} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix},$$

solves (4.5) into (4.4), as claimed. \square

This theorem gives the algorithm for retrieving the structure of the characteristic tree from the harmonic matrix a of a given graph.

5 Harmonic Loops

Now we shall look for the structure of the loop ensemble of a given graph. The elements of group $L(a)$ act — if restricted to the subspace $L(G) \in \mathbb{V}_G$ — as automorphisms. The action of $L(a)$ decomposes the subspace $L(G)$ into a number of orbits. Since the group $L(a)$ is cyclic, the orbits are loops. (Clearly, the orbits do not form linear subspaces of $L(G)$). It follows thus:

Proposition 5.1 *The least common multiple of the lengths of the loops in $L(G)$ is equal to the order of the loop group, $m = |L(a)|$. There is a loop in $L(G)$ of length m . In particular, the rank of any vector of $L(G)$ divides m .*

Denote a loop of length i by L_i . If nL_i denotes a disjoint sum of n loops of order i , then the decomposition of $L(G)$ into loops may be expressed symbolically as a formal sum

$$L(G) = n_1L_1 + n_2L_2 + \dots + n_mL_m \quad (5.1)$$

(for examples this notation see Fig. 5) Due to Proposition 5.1, the only nonzero numbers n_i could be those with i dividing $m = |L(a)|$. The total number of loops is $n = \sum n_i$. Clearly all lengths of the loops sum up to a power of two, namely $\sum_i i \cdot n_i = |L(G)| = 2^{\dim L(G)}$.

Now we recover the exact structure of the loop ensemble. That is, we retrieve the coefficients of the decomposition (5.1), from the monoid $M(a)$. We shall denote the eigenspace of an endomorphism g by $F(g)$, and its dimension by $\dim F(g)$.

Theorem 5.2 *The number of loops of length p in $L(G)$ is*

$$n_p = \frac{1}{p} \sum_{i|p} \mu\left(\frac{p}{i}\right) 2^{\dim F(a^i)} \quad (5.2)$$

where μ denotes Moebius function, and the sum runs over divisors of p .

Proof: First, notice that $F(a^i) \subset F(a^j)$ only if $i|j$. Hence, for any j we have

$$|F(a^j)| = \sum_{d|j} d \cdot n_d = \sum_{d=1}^j d \cdot n_d \cdot \mu(d, j) \quad (5.3)$$

where μ is the “factorization function” defined: $\mu(i, j) = 1$ if $i|j$, and $\mu(i, j) = 0$ otherwise. For conciseness, denote $F_i = |F(a^i)| = 2^{\dim F(a^i)}$ (this is the number

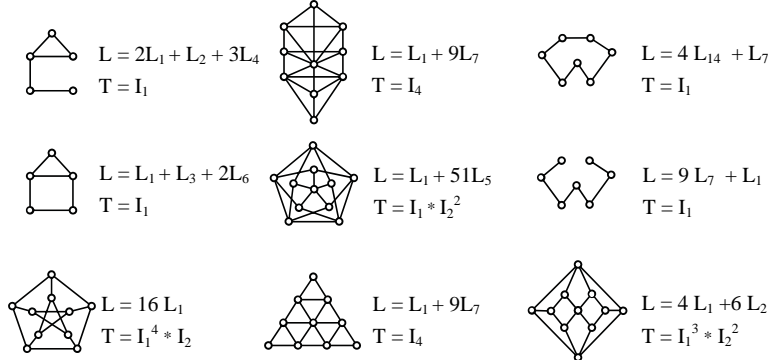


Figure 7: Examples of graphs and their loop and tree structure.

of states in the eigenspace of a^i). The system of equations (5.3) has form $\mathbf{F} = \mathbf{A}\mathbf{n}$, where A is a lower-triangular matrix

$$A = \begin{bmatrix} 1 & & & & & & & & \\ 1 & 2 & & & & & & & \\ 1 & 0 & 3 & & & & & & \\ 1 & 2 & 0 & 4 & & & & & \\ 1 & 0 & 0 & 0 & 5 & & & & \\ 1 & 2 & 3 & 0 & 0 & 6 & & & \\ & & & & & & \ddots & & \end{bmatrix} \quad \text{or} \quad A_{ij} = i \cdot \mu(i, j) \quad (5.4)$$

It is well-known that the inverse of matrix $M_{ij} = \mu(i, j)$ is given by $(M^{-1})_{ij} = \mu(\frac{i}{j})$, where μ is the Moebius function, defined

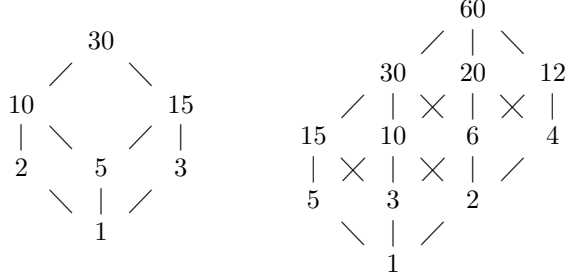
$$\mu(d) = \begin{cases} 1 & \text{if } d = 1 \\ (-1)^r & \text{if } d = p_1 \cdot p_2 \cdot \dots \cdot p_r \\ 0 & \text{if } p^2 | d \text{ for any prime } p \end{cases}$$

where p_1, p_2, \dots, p_r are distinct primes (for the properties of the Moebius function see [4]). Equation (5.4) is a modified version of the factorization function, and it easily solves to (5.2). \square

The above theorem provides an algorithm to construct the structure of the loop ensemble of the evolution digraph of a given graph.

Remark 5.3 Here is alternative view on the inversion (5.2) of the system (5.3). Let $\{\mathbf{N}, \prec\}$ be the poset (lattice) of natural numbers with the order induced by multiplicative relations, i.e. $a \prec b$ if $a|b$. Let $[a]$ denote the sub-poset of all elements below a , i.e. $[a] = \{n \in \mathbf{N} \mid n \prec a\}$ (a filter generated by a). Let \bar{a} be the set of maximal elements in $[a] - \{a\}$; say $\bar{a} = \{a_1, \dots, a_n\}$. The subsets $[a_1]$,

$[a_2], \dots, [a_n]$ form a “ \cap -algebra”, i.e. their mutual intersections are again some subposets generated by elements of $[a]$.



The numbers of vertices in loops of order k may be obtained by the inclusion-exclusion principle applied to the system of subsets generated by \bar{k} . For example: $n_{30} = F_{30} - F_{10} - F_{15} + F_5$ and $n_{60} = F_{60} - F_{30} - F_{20} - F_{12} + F_{10} + F_6 + F_4 - F_2$.

6 Disjoint sum of graphs

The results of the previous sections may be formulated as follows. On the one hand we have a category \mathcal{G} of finite graphs. On the other we have the category of binary-generated trees \mathcal{T} and the category of loop ensembles \mathcal{L} :

$$\begin{aligned} \mathcal{T} &= \{ \prod_i I_i^{n_i} \mid i, n_i \in \mathbf{N} \} \\ \mathcal{L} &= \{ \sum_i n_i L_i \mid i, n_i \in \mathbf{N} \} \end{aligned} \tag{6.1}$$

The construction described in this paper is a map

$$* : \mathcal{G} \longrightarrow \mathcal{L} \times \mathcal{T} \tag{6.2}$$

which associates to each graph G two elements, $L(G) \in \mathcal{L}$ and $T(G) \in \mathcal{T}$. The digraph of evolution G^* is a semidirect product of these two, namely, if $\{L, \varphi\}$ and $\{T, \psi\}$ are viewed as digraphs

$$G^* = \{L \times T, \varphi\psi\} = L \times T$$

The problem is to find how various operations and maps in the category of graphs are reflected in the algebraic properties of the spaces \mathcal{L} and \mathcal{T} .

From many such problems we answer a modest question concerning the evolution graph for a disjoint sum of two graphs, $G_1 \cup G_2$. Before we go on, a few remarks on the two categories: of trees and of loop ensembles.

Universal family of binary generated trees. The family of trees $\mathcal{T} := \text{gen} \{I_0, I_1, I_2, \dots\}$ consists of all finite $*$ -products of binomial trees. It forms a semigroup $(\mathcal{T}, *)$ satisfying:

$$\begin{aligned}
1) \quad & T_1 * T_2 = T_2 * T_1 \\
2) \quad & (T_1 * T_2) * T_3 = T_1 * (T_2 * T_3) \\
3) \quad & T * I_0 = T
\end{aligned} \tag{6.3}$$

Thus $(\mathcal{T}, *)$ forms a commutative free monoid with exponentials from \mathbf{N} .

Universal loop space. Similarly, let us introduce a “space” \mathcal{L} over \mathbf{N} spanned by the the set $\{L_1, L_2, \dots\}$, where L_i represents a loop length i . Using additive notation, a collection of loops (loop ensemble) will be denoted as a formal sum, e.g. $L = L_3 + 2L_4 = \{\triangle, \square, \square\}$. Now we shall turn \mathcal{L} into an algebra by introducing a product $*$: $\mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$, which for single loops L_a and L_b of length a and b respectively is defined

$$L_a * L_b = \langle a|b \rangle L_{[a|b]} \tag{6.4}$$

where we denote $\langle a|b \rangle = \gcd(a, b)$ and $[a|b] = \text{lcm}(a, b)$.

Theorem 6.1 *The triple $\{\mathcal{L}, +, *\}$ satisfies*

$$\begin{aligned}
1) \quad & L_a * L_b = L_b * L_a & 1') \quad & L_a + L_b = L_b + L_a \\
2) \quad & (L_a * L_b) * L_c = L_a * (L_b * L_c) & 2') \quad & (L_a + L_b) + L_c = L_a + (L_b + L_c) \\
3) \quad & L_1 * L = L \text{ for any } L & 3') \quad & L_0 + L = L \text{ for any } L \\
4) \quad & (L_a + L_b) * L_c = L_a * L_c + L_b * L_c
\end{aligned}$$

Proof: Let us show item 2): Recall that $\langle a|b \rangle = \frac{a \cdot b}{[a|b]}$. Using (6.4) we get $(L_a * L_b) * L_c = \frac{a \cdot b}{[a|b]} L_{[a|b]} * L_c = \frac{a \cdot b}{[a|b]} \cdot \frac{[a|b] \cdot c}{[a|b|c]} L_{[a|b|c]} = \frac{a \cdot b \cdot c}{[a|b|c]} L_{[a|b|c]} = \langle a|b|c \rangle L_{[a|b|c]}$. Since $[\ | \]$ is an associative operation, property (2) holds. \square

The definition in Eq. 6.4 is chosen to make the following look simple.

Proposition 6.2 *The evolution digraph of a disjoint union of two graphs is*

$$(G_1 + G_2)^* = \{T(G_1) * T(G_2), L(G_1) * L(G_2)\} \tag{6.5}$$

where the star operations are defined in (6.4) for loops, and (6.4) for the characteristic trees.

7 Power Graphs

Definition 7.1 *The p^{th} power of a graph G is the graph G^p determined by the p^{th} power of the harmonic matrix of graph G , a^p .*

Clearly, G^p has the same set of vertices as G , but a different structure of edges. That the powers of a graph are well-defined is obvious, since $a^p \Omega = a^{p-1}(a\Omega) = 0$ and $(a^p)^T = (a^T)^p = a^p$. An interesting question is to determine the evolution digraph of graph G^p from the the evolution digraph of G alone.

From the basic properties of the monoid $M(a)$ we obtain the length of the tail and rank of the cycle group of $M(a^p)$ to be

$$\begin{aligned} \text{tail:} & \quad |T(a^p)| = \text{Ent}(|T(a)|/q) \\ \text{loop:} & \quad |L(a^p)| = |L(a)| // q, \end{aligned} \quad (7.1)$$

where $//$ denotes “funny division,” defined for integers by $a//b = [a|b]/b$ (recall that $[a|b] = \text{lcm}(a, b)$). For instance $6//1 = 6$, $6//2 = 3$, $6//3 = 2$, $6//4 = 3$, $6//5 = 6$, $6//6 = 1$.

Proposition 7.2 *Let the evolution digraph G^* of graph G have the following characteristic tree and the loop ensemble:*

$$\begin{aligned} T(G) &= I_1^{b_1} * I_2^{b_2} * \dots * I_k^{b_k} = \prod_{i=1}^k I_i^{b_i} \\ L(G) &= c_1 L_1 + c_2 L_2 + \dots + c_m L_m = \sum_{i=1}^m c_i L_i, \end{aligned}$$

where $k = |T(a)|$ and $m = |L(a)|$. Then the evolution digraph $(G^q)^*$ consists of

$$\begin{aligned} T(G^q) &= \prod_{i=1}^{k'} I_i^{w_i} \quad \text{where} \quad w_i = \sum_{j=q(i-1)+1}^{q \cdot i} j \cdot b_j \\ L(G^q) &= \sum_{i=1}^{m'} c_i \cdot \langle i|q \rangle L_{i//q} \end{aligned} \quad (7.2)$$

where $k' = |T(a^p)|$ and $m' = |L(a^p)|$ are those of (7.1) above.

Proof: The proof of this is simpler in combinatorial terms. Express G^* as the product of the loop ensemble and the characteristic tree. A state S is equivalent to a choice of a node in one of the loops in $L(G)$, and a node in each binary tree constituting $T(G)$ (imagine them as “glowing points”). The action of the harmonic operator on S is represented now by a shift of each glowing point by one position: along the loop, and down each tree. Tracing their q -step movements brings the conclusion to the proof. \square

Remark: Every integer $q \in \mathbf{N}$ defines a linear map on the universal loop space $\hat{q}: \mathcal{L} \rightarrow \mathcal{L}$ which transforms each single loop L_a into a sum of loops according to

$$L_a \rightarrow \langle a|q \rangle L_{a//q}$$

as suggested by (7.2). For instance $\hat{4}(L_6) = 2L_3$, which is due to the *two* distinct paths of “4-step jumps” in a hexagon. The second part of Proposition 7.2 may be now expressed simply

$$L(G^q) = \hat{q}L(G)$$

Notice that operator $\hat{\cdot}$ is a representation of natural numbers with multiplication, namely $\widehat{ab} = \widehat{a} \cdot \widehat{b}$. On the other hand, we have the *adjoint* representation of the algebra $\{\mathcal{L}, *\}$, where for $a \in \mathbf{N}$ we define $\overset{*}{a} L_i = L_a * L_i$. Each of these

representations is of course commutative, but the common algebra of both kinds of operators is not. For instance:

$${}^* \hat{2} L_i = \begin{cases} L_{2i} & \text{if } i = \text{odd} \\ 2L_i & \text{if } 2|i \text{ but } 4 \nmid i \\ 4L_{i/2} & \text{if } 4|i \end{cases} \quad \text{whereas} \quad \hat{2} {}^* L_i = \begin{cases} 2L_i & \text{if } i = \text{odd} \\ 4L_{i/2} & \text{if } i = \text{even} \end{cases}$$

8 Conclusion

We defined the harmonic operator of a graph as a topologically motivated endomorphism of the space of states of graphs. The states (co-states) are meant as vectors in the linear space over \mathbf{Z} formally spanned by the vertices (edges). The paper concentrates on the former. We defined a dynamical system on a graph as a recursive application of the harmonic operator. We found the types of evolutions (evolution digraphs) and classified them. An algorithm to foretell the evolution digraph for a particular graph has consists of Theorems 4.2 and 5.2.

We see it as a map

$$\mathcal{G} \longrightarrow \mathcal{L} \times \mathcal{T} \tag{8.1}$$

which associates to each graph G two vectors, $L(G) \in \mathcal{L}$ and $T(G) \in \mathcal{T}$. Trees and loops correspond to structure of invariant subspaces of a ; any number derived from it may be viewed as an invariant of the graph.

Map (8.1) is a morphism of objects, and the correspondence for operations of the disjoint union and for the graph power were shown. The problem of finding such a correspondence for other operations on graphs may prove to be rather difficult.

The inverse problem asks whether a given digraph represents a harmonic evolution of some graph. Equivalently, the problem is to characterize the images $L(\mathcal{G}) \subset \mathcal{L}$ and $T(\mathcal{G}) \subset \mathcal{T}$. We offer these two conjectures.

Conjecture 1: *Any finite product of binary trees of height at least 1 is a characteristic tree for some graph.*

Conjecture 2: *Denote $\mathcal{H} = \{2^i \mid i \in \mathbf{N}\} = \{1, 2, 4, 8, \dots\}$ (the set of “highly even numbers”). A formal sum $L = \sum n_i L_i$ is said to be admissible, $L \in \mathcal{L}_A$, if the following is satisfied:*

- 1) $n_1 \in \mathcal{H}$
- 2) $\sum_{i|p} i \cdot n_i \in \mathcal{H}$ for each p such that $n_p \neq 0$
- 3) if $n_i \neq 0$ or $n_j \neq 0$, then $n_{[i|j]} \neq 0$
- 4) $\log_2 \sum_i i \cdot n_i$ is even

where we denote $[i|j] = \text{l.c.m.}(i, j)$. Then the set of harmonic loop ensembles of finite graphs coincides with the above admissible set, $\mathcal{L}(\mathcal{G}) = \mathcal{L}_A$.

Further exercises and problems: 1. rewrite the theory for the co-harmonic operator $b = \delta\partial$ acting on the edges and defining the harmonic "co-evolution" on graphs. Can one determine the corresponding co-evolution digraph from the evolution graph alone? 2. Replace the field \mathbf{Z}_2 by another commutative ring (like \mathbf{Z}_3). 3. Classify the rules for evolution on graphs other than harmonic rules.

Acknowledgements

The author would like to thank Philip Feinsilver and Greg Budzban for fruitful conversations and for their interest in this project.

References

- [1] W. N. Anderson, Jr. and TD Morley, Eigenvalues of the Laplacian of a graph, *Linear and Multilinear Algebra* **18** (1985) 141–145.
- [2] Chris Godsil and Gordon Royle, *Algebraic Graph Theory*, Springer, 2000.
- [3] Erich Prisner, *Graph Dynamics*. (Longman, Essex, 1995).
- [4] Gian-Carlo Rota, *On the Foundations of Combinatorial Theory, I. Theory of Moebius Function*, *Z. Wahrscheinlichkeitstheorie* **2** (1964) 340–368.
- [5] W. T. Tutte, *Graph Theory* (Addison-Wesley, Reading, 1984) (as Volume 21 of *Encyclopedia of Mathematics and its Applications*, G.-C. Rota, ed.).
- [6] John Archibald Wheeler, *Frontiers of Time* (North-Holland, Amsterdam, 1979).
- [7] John Archibald Wheeler, *Pregeometry: Motivations and prospects*. In *Quantum theory and Gravitation* (Springer, 1980).