Apollonian Circle Packings: Number Theory

Ronald L. Graham [1]
Jeffrey C. Lagarias [2]
Colin L. Mallows
Allan R. Wilks
AT&T Labs, Florham Park, NJ 07932-0971

Catherine H. Yan
Texas A&M University, College Station, TX 77843

(November 1, 2002 version)

ABSTRACT

Apollonian circle packings arise by repeatedly filling the interstices between mutually tangent circles with further tangent circles. It is possible for every circle in such a packing to have integer radius of curvature, and we call such a packing an *integral Apollonian circle packing.* This paper studies number-theoretic properties of the set of integer curvatures appearing in such packings. Each Descartes quadruple of four tangent circles in the packing gives an integer solution to the Descartes equation, which relates the radii of curvature of four mutually tangent circles: $x^2+y^2+z^2+w^2 = \frac{1}{2}(x+y+z+w)^2$. Each integral Apollonian circle packing is classified by a certain *root quadruple* of integers that satisfies the Descartes equation, and that corresponds to a particular quadruple of circles appearing in the packing. We express the number of root quadruples with fixed minimal element $-n$ as a class number, and give an exact formula for it. We study which integers occur in a given integer packing, and determine congruence restrictions which sometimes apply. We present evidence suggesting that the set of integer radii of curvatures that appear in an integral Apollonian circle packing has positive density, and in fact represents all sufficiently large integers not excluded by congruence conditions. Finally, we discuss asymptotic properties of the set of curvatures obtained as the packing is recursively constructed from a root quadruple.

Keywords: Circle packings, Apollonian circles, Diophantine equations

[1]Current address: Dept. of Computer Science, Univ. of Calif. at San Diego, La Jolla, CA 92093
[2]Work partly done during a visit to the Institute for Advanced Study.

Apollonian Circle Packings: Number Theory

1. Introduction

Place two tangent circles of radius 1/2 inside and tangent to a circle of radius 1. In the two resulting curvilinear triangles fit tangent circles as large as possible. Repeat this process for the six new curvilinear triangles, and so on. The result is pictured in Figure 1, where each circle has been labeled with its curvature—the reciprocal of its radius.

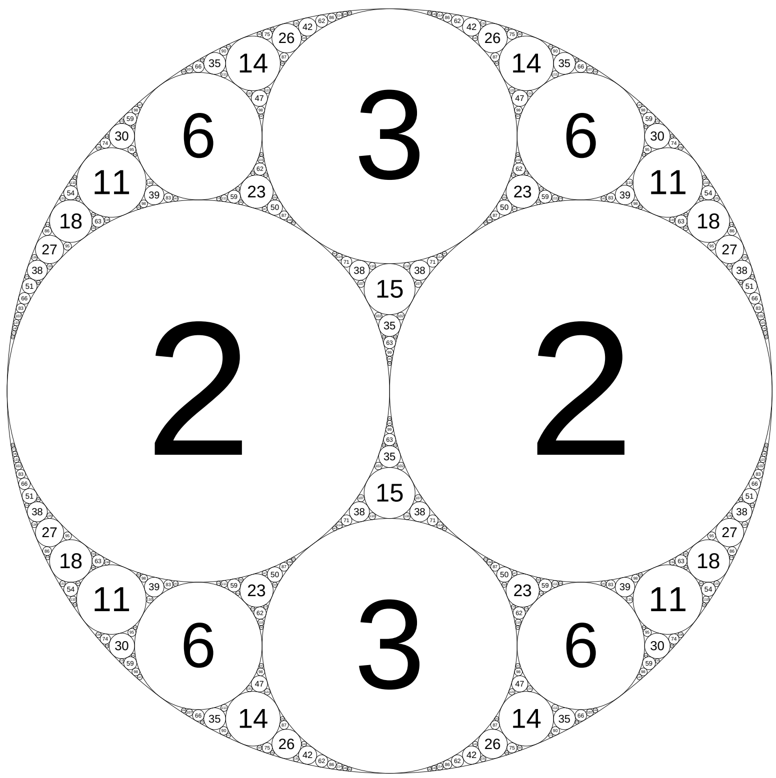

Figure 1: The integral Apollonian circle packing (-1, 2, 2, 3)

Remarkably, every circle in Figure 1 has integer curvature. Even more remarkable is that if the picture is centered at the origin of the Euclidean plane with the centers of the "2" circles on the x-axis, then each circle in the picture has the property that the coordinates of its center, multiplied by its curvature, are also integers. In this paper we are concerned with circle packings having the first of these properties; the latter property is addressed in a companion paper [18, Section 3].

An *Apollonian circle packing* is any packing of circles constructed recursively from an initial configuration of four mutually tangent circles by the procedure above. More precisely, one starts from a *Descartes configuration*, which is a set of four mutually tangent circles with disjoint interiors, suitably defined. In the example above, the enclosing circle has "interior" equal to its exterior, and its curvature is given a negative sign. Recall that in a quadruple of mutually touching circles the curvatures $(a,\ b,\ c,\ d)$ satisfy the *Descartes equation*

$$a^2 + b^2 + c^2 + d^2 = \frac{1}{2}(a+b+c+d)^2, \tag{1.1}$$

as observed by Descartes in 1643 (in an equivalent form). Any quadruple $(a,\ b,\ c,\ d)$ satisfying this equation is called a *Descartes quadruple.* An *integral Apollonian circle packing* is an Apollonian circle packing in which every circle has an integer curvature. The starting point of this paper is the observation that if an initial Descartes configuration has all integral curvatures, then the whole packing is integral, and conversely. This integrality property of packings has been discovered repeatedly; perhaps the first observation of it is in the 1937 note of F. Soddy [45] "The bowl of integers and the Hexlet". It is discussed in some detail in Aharonov and Stephenson [1].

In this paper we study integral Apollonian circle packings viewed as equivalent under Euclidean motions, an operation which preserves the curvatures of all circles. Such packings are classified by their root quadruple, a notion defined in §3. This is the "smallest" quadruple in the packing as measured in terms of curvatures of the circles. In the packing above the root quadruple is $(-1,\ 2,\ 2,\ 3)$, where -1 represents the (negative) curvature of the bounding circle. We study the set of integers (curvatures) represented by a packing using the *Apollonian group* $\mathcal{A}$, which is a subgroup of $GL(4,\mathbb{Z})$ which acts on integer Descartes quadruples. This action permits one to "walk around" on a fixed Apollonian packing, moving from one Descartes quadruple to any other quadruple in the same packing. The Apollonian group was introduced by Hirst [21] in 1967, who used it bounding the Hausdorff dimension of the residual set of an Apollonian packing; it was also used in Söderberg,[46] and Aharonov and Stephenson [1]. Descartes quadruples associated to different root quadruples cannot be reached by the action of $\mathcal{A}$, and the action of the Apollonian group partitions the set of integer Descartes quadruples into infinitely many equivalence classes (according to which integral Apollonian packing they belong.) By scaling an integer Apollonian packing by an appropriate homothety, one may obtain a *primitive integral Apollonian packing,* which is one whose Descartes quadruples have integer curvatures with greatest common divisor 1. Thus the study of integral Apollonian packings essentially reduces to the study of primitive packings.

The simplest integral Apollonian circle packing is the one with root quadruple $(0,\ 0,\ 1,\ 1)$, which is pictured in Figure 2. This packing is special in several ways. It is degenerate in that it has two circles with “center at infinity”, whose boundaries are straight lines, and it is the only primitive integral Apollonian circle packing that is unbounded. It is also the only primitive integral Apollonian circle packing that contains infinitely many copies of the root quadruple. This particular packing has already played a role in number theory. That part of the packing in an interval of length two between the tangencies of two adjacent circles of radius one, consisting of the (infinite) set of circles tangent to one of the straight lines, forms a set of “Ford circles”, after shrinking all circles by a factor of two. These circles, introduced by Ford (see [15]), can be labelled by the Farey fractions on the interval $(0, 1)$ and used to prove basic results in one-dimensional Diophantine approximation connected with the Markoff spectrum, see Rademacher [38, pp. 41–46] and Nicholls [36].

In this paper our interest is in those properties of the set of integral Apollonian circle packings that are of a Diophantine nature. These include the distribution of integer Descartes quadruples, of integer root quadruples, and the representation and the distribution of the integers (curvatures) occurring in a fixed integral Apollonian circle packing. Finally we consider the size distribution of elements in the Apollonian group, a group of integer matrices associated to such packings.

To begin with, the full set of all integer Descartes quadruples (taken over all integral Apollonian packings) is enumerated by the integer solutions to the Descartes equation (up to a sign.) In §2 we determine asymptotics for the total number of integer solutions to the Descartes equation of Euclidean norm below a given bound.

In §3 we define the Apollonian group. We describe a reduction theory which multiplies Descartes quadruples by elements of this group and uses it to find a quadruple of smallest size in a given packing, called a *root quadruple.* We prove the existence and uniqueness of a root quadruple associated to each integral Apollonian packing.

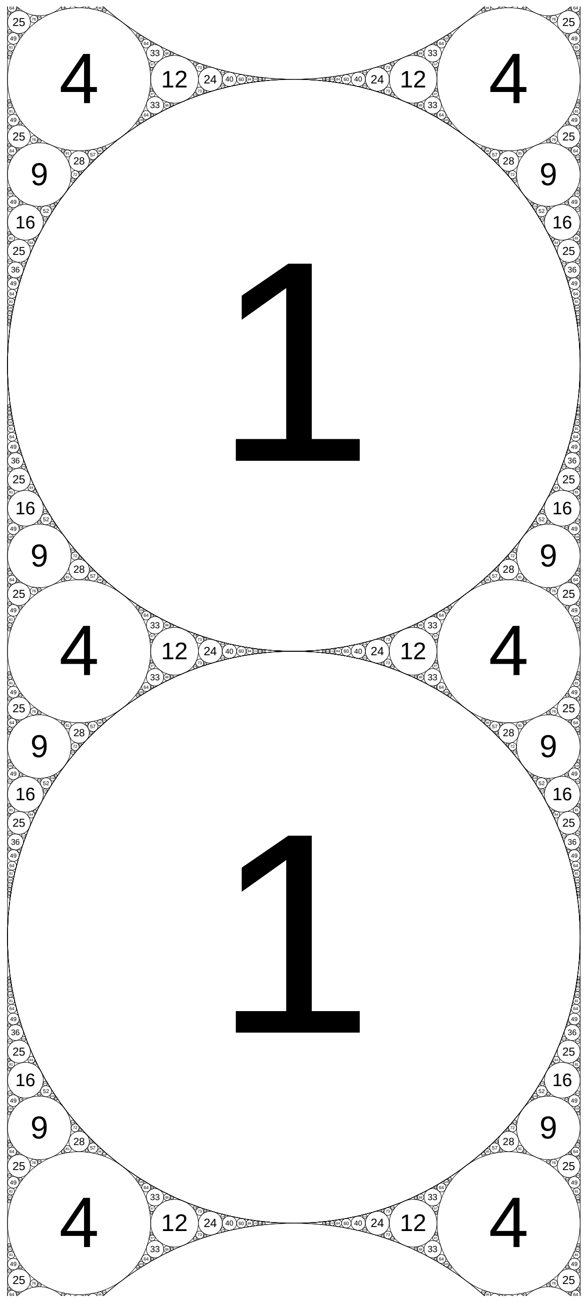

Figure 2: The Apollonian circle packing (0, 0, 1, 1).

In §4 we study the root quadruples of primitive integer packings. We show that the number $N^*_{root}(-n)$ of primitive root quadruples containing a given negative integer $-n$ is equal to the class number of primitive integral binary quadratic forms of discriminant $-4n^2$, under $GL(2,\mathbb{Z})$-equivalence. Using this fact we obtain in Theorem 4.3 an exact formula for the number of such quadruples, which is

$$N^*_{root}(-n) = \frac{n}{4}\prod_{p|n}(1 - \frac{\chi_{-4}(p)}{p}) + 2^{\omega(n)-\delta_n-1}, \tag{1.2}$$

where $\chi_{-4}(n) = (-1)^{(n-1)/2}$ for n and 0 for even n, $\omega(n)$ denotes the number of distinct primes dividing n, and $\delta_n = 1$ if $n \equiv 2 (\bmod\ 4)$ and $\delta_n = 0$ otherwise. This formula was independently discovered and proved by S. Northshield [37]. As a consequence we show that $N^*_{root}(-n)$ has an upper bound $O(n \log\log n)$ and a lower bound $\Omega(\frac{n}{(\log\log n)})$, respectively. These bounds are sharp up to multiplicative constants.

In §5 we study the integer curvatures appearing in a single integral Apollonian packing, counting integers with multiplicity. D. Boyd [7] showed that the number of circles occurring in a bounded Apollonian packing having curvature less than a bound T grows like $T^{\alpha+o(1)}$, where $\alpha \approx 1.30...$ is the Hausdorff dimension of the residual set of any Apollonian circle packing. This result applies to integral Apollonian packings. We observe that these integers can be put in one-to-one correspondence with elements of the Apollonian group, using the root quadruple. Using this result we show that the number of elements of the Apollonian group which have norm less than T is of order $T^{\alpha+o(1)}$, as $T \to \infty$.

In §6 we study the integer curvatures appearing in a packing, counted without multiplicity. We show that there are always nontrivial congruence restrictions (*mod* 24) on the integers that occur. We give some evidence suggesting that such congruence restrictions can only involve powers of the primes 2 and 3. We conjecture that in any integral Apollonian packing, all sufficiently large integers occur, provided they are not excluded by a congruence condition. This may be a hard problem, however, since we show that it is analogous to Zaremba's conjecture stating that there is a fixed integer K such that for all denominators $n \geq 2$ there is a rational $\frac{a}{n}$ in lowest terms whose continued fraction expansion has all partial quotients bounded by K.

In §7 we study the set of integer curvatures at "depth n" in an integral Apollonian packing, where n measures the distance to the root quadruple. There are exactly $4 \times 3^{n-1}$ such elements.

We determine the maximal and minimal curvature in this set, and also formulate a conjecture concerning the asymptotic size of the median curvature as $n \to \infty$. These problems are related to the joint spectral radius of the matrix generators $\Sigma = \{S_1, S_2, S_3, S_4\}$ of the Apollonian group, which we determine.

In §8 we conclude with some directions for further work and a list of open problems.

There has been extensive previous work on various aspects of Apollonian circle packings, related to geometry, group theory and fractals. The name "Apollonian packing" traces back to Kasner and Supnick [23] in 1943. It has been popularized by Mandlebrot [29, p. 169ff], who observed a connection with work of Apollonius of Perga, around 200BC, see also Coxeter [8]. Further discussion and references can be found in Aharonov and Stephenson [1] and Wilker [52]. See also the companion papers [17], [18],[19] and [24], where we investigate a variety of group-theoretic properties of these configurations, as well as various extensions to higher dimensions and other spaces, such as hyperbolic space.

Notation. We use several different measures of the size of a vector $\mathbf{w} = (w_1, w_2, w_3, w_4)$. In particular we set $L(\mathbf{w}) := w_1 + w_2 + w_3 + w_4$, a measure of size which is not a norm. We let $H(\mathbf{w}) := (w_1^2 + w_2^2 + w_3^2 + w_4^2)^{1/2}$ or $|\mathbf{w}|$ denote the Euclidean norm, while $||\mathbf{w}||_\infty := \max_{1 \le i \le 4} |w_i|$ denotes the supremum norm of $\mathbf{w}$.

2. Integral Descartes Quadruples

An Apollonian circle packing is *integral* if every circle of the packing has an integer curvature. From (1.1) it follows that if $a, b, c,$ are given, the curvatures d, d' of the two circles that are tangent to all three satisfy

$$d,\ d' = a + b + c \pm 2q_{abc},$$

where

$$q_{abc} = \sqrt{ab + bc + ac}.$$

Hence

$$d + d' = 2(a + b + c). \tag{2.1}$$

In other words, given four mutually tangent circles with curvatures a, b, c, d, the curvature of the other circle that touches the first three is given by

$$d' = 2a + 2b + 2c - d. \tag{2.2}$$

It follows that an Apollonian packing is integral if the starting Descartes quadruple consists entirely of integers.

The relation (2.1) is the basis of the integrality property of Apollonian packings. It generalizes to n dimensions, where the curvatures X_i of a set of $n+1$ mutually tangent spheres in $\mathbb{R}^n$ (having distinct tangents) are related to the curvatures X_0 and X_{n+2} of the two spheres that are tangent of all of these by

$$X_0 + X_{n+2} = \frac{2}{n-1}(X_1 + X_2 + \cdots + X_n).$$

This relation gives integrality in dimensions $n = 2$ and $n = 3$; the three dimensional case is studied in Boyd [5]. It even generalizes further to sets of equally inclined spheres with inclination parameter γ, with the constant $\frac{2}{n+\frac{1}{\gamma}}$; the case $\gamma = -1$ is the mutually tangent case, cf. Mauldon [31] and Weiss [50, Theorem 3].

Definition 2.1. (i) An *integer Descartes quadruple* $\mathbf{a} = (a, b, c, d) \in \mathbb{Z}^4$ is any integer representation of zero by the indefinite integral quaternary quadratic form,

$$Q_{\mathcal{D}}(w, x, y, z) := 2(w^2 + x^2 + y^2 + z^2) - (w + x + y + z)^2,$$

which we call the *Descartes integral form.* That is, writing $\mathbf{v} = (w,\ x,\ y,\ z)^T$, we have $Q_{\mathcal{D}}(w, x, y, z) = \mathbf{v}^T Q_{\mathcal{D}} \mathbf{v}$, where

$$Q_{\mathcal{D}} = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}. \tag{2.3}$$

This quadratic form has determinant $det(Q_{\mathcal{D}}) = -16$ and, on identifying the form $Q_{\mathcal{D}}$ with its symmetric integral matrix, it satisfies $Q_{\mathcal{D}}^2 = 4I$.

(ii) An integer Descartes quadruple is *primitive* if $gcd(a, b, c, d) = 1$.

In studying the geometry of Apollonian packings ([17]-[19], [24]) we use instead a scaled version of the Descartes integral form, namely the *Descartes quadratic form* $Q_2 := \frac{1}{2} Q_{\mathcal{D}}$.

Definition 2.2. The size of any real quadruple $(a, b, c, d) \in \mathbb{R}^4$ is measured by the *Euclidean height* $H(\mathbf{a})$, which is:

$$H(\mathbf{a}) := (a^2 + b^2 + c^2 + d^2)^{1/2}. \tag{2.4}$$

Now let $N_{\mathcal{D}}(T)$ count the number of integer Descartes quadruples with Euclidean height at most T. We shall relate this quantity to the number $N_{\mathcal{L}}(T)$ of integer Lorentz quadruples of height at most T, where *Lorentz quadruples* are those quadruples that satisfy the Lorentz equation

$$-W^2 + X^2 + Y^2 + Z^2 = 0. \tag{2.5}$$

These are the zero vectors of the *Lorentz quadratic form*

$$Q_{\mathcal{L}}(W, X, Y, Z) = -W^2 + X^2 + Y^2 + Z^2, \tag{2.6}$$

whose matrix representation is

$$Q_{\mathcal{L}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Similarly we shall relate the number of primitive integer Descartes quadruples, denoted $N^*_{\mathcal{D}}(T)$, to the number of primitive integer Lorentz quadruples of height at most T, denoted $N^*_{\mathcal{L}}(T)$. We show that there is a one-to-one height preserving correspondence between integer Descartes quadruples and integer Lorentzian quadruples. Introduce the matrix J_0 defined by

$$J_0 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{2.7}$$

and note that ${J_0}^2 = I$. The Descartes and Lorentz forms are related by

$$Q_{\mathcal{D}} = 2 J_0^T Q_{\mathcal{L}} J_0, \tag{2.8}$$

which leads to a relation between their zero vectors.

Lemma 2.1. *The mapping* $(W,\ X,\ Y,\ Z)^T = J_0(w,\ x,\ y,\ z)^T$ *gives a bijection from the set* $(w,\ x,\ y,\ z)$ *of real Descartes quadruples to that of real Lorentz quadruples* $(W,\ X,\ Y,\ Z)$ *which preserves height. It restricts to a bijection from the set of integer Descartes quadruples to integer Lorentz quadruples, so that* $N_{\mathcal{D}}(T) = N_{\mathcal{L}}(T)$*, for all* $T > 0$*, and from primitive integer Descartes quadruples to primitive integer Lorentz quadruples, so that* $N^*_{\mathcal{D}}(T) = N^*_{\mathcal{L}}(T)$*, for all* $T > 0$.

Proof. An easy calculation shows that the mapping takes real solutions of one equation to solutions of the other and that the inverse mapping is $(w,\ x,\ y,\ z)^T = J_0(W,\ X,\ Y,\ Z)^T$, so that it is a bijection. The mapping takes integer Descartes quadruples to integer Lorentz quadruples because any integer solution to the Descartes equation satisfies $w+x+y+z \equiv 0\ (mod\ 2)$. This also holds in the reverse direction because integer solutions to the Lorentz form also satisfy $W + X + Y + Z \equiv 0\ (mod\ 2)$, as follows by reducing (2.5) $(mod\ 2)$. It is easy to check that primitive integer Descartes quadruples correspond to primitive integer Lorentz quadruples. □

Counting the number of integer Descartes quadruples of height below a given bound T is the same as counting integer Lorentz quadruples. This is a special case of the classical problem of estimating the number of representations of a fixed integer by a fixed diagonal quadratic form, on which there is an enormous literature. For example Ratcliffe and Tschantz [40] give asymptotic estimates with good error terms for the number of solutions for the equation $X^2 + Y^2 + Z^2 - W^2 = k$, of Euclidean height below a given bound, for all $k \neq 0$. (They treat Lorentzian forms in n variables.) Rather surprisingly the case $k = 0$ seems not to be treated in the published literature, so we derive an asymptotic formula with error term below. The main term in this asymptotic formula was found in 1993 by W. Duke [13](unpublished) in the course of establishing an equidistribution result for its solutions.

Theorem 2.2. *The number of integer Descartes quadruples* $N_{\mathcal{D}}(T)$ *of Euclidean height at most* T *satisfies* $N_{\mathcal{D}}(T) = N_{\mathcal{L}}(T)$*, and*

$$N_{\mathcal{L}}(T) = \frac{\pi^2}{4L(2, \chi_{-4})}\ T^2 + O(T(logT)^2), \tag{2.9}$$

as $T \to \infty$*, in which*

$$L(2, \chi_{-4}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.9159.$$

The number $N^*_{\mathcal{D}}(T)$ of primitive integer Apollonian quadruples of Euclidean height less than T satisfies $N^*_{\mathcal{D}}(T) = N^*_{\mathcal{L}}(T)$ and

$$N^*_{\mathcal{L}}(T) = \frac{3}{2L(2,\chi_{-4})}\ T^2 + O(T(logT)^2), \tag{2.10}$$

as $T \to \infty$.

Proof. By Lemma 2.1 it suffices to estimate $N_{\mathcal{L}}(T)$. Let $r_3(m)$ denote the number of integer representations of m as a sum of three squares, allowing positive, negative and zero integers. Rewriting the Lorentz equation as $X^2 + Y^2 + Z^2 = W^2$ we obtain for integer T that

$$N_{\mathcal{L}}(\sqrt{2}T) = 1 + 2\sum_{m=1}^{T} r_3(m^2), \tag{2.11}$$

since there are two choices for W whenever $W \neq 0$. A general form for $r_3(m)$ was obtained in 1801 by Gauss [16, Articles 291-292], while in the special case $r_3(m^2)$ a simpler form holds, given in 1906 by Hurwitz [22]. This is reformulated in Sandham [43, p. 231] in the form: if $m = \prod_p p^{e_p(m)}$, and p runs over the primes and $m_{odd} = m2^{-e_2(m)}$, then

$$\begin{aligned} r_3(m^2) &= 6m_{odd} \prod_{p\equiv 3 \ (mod\ 4)} (1 + \frac{2}{p} + ... + \frac{2}{p^{e_p(m)}}) \\ &= 6 \prod_{p \equiv 1\ (mod\ 2)} \left(\frac{p^{e_p(m)+1} - 1 - (\frac{-4}{p})(p^{e_p(m)} - 1)}{p-1} \right). \end{aligned} \tag{2.12}$$

Sandham observes that this formula is equivalent to

$$F(s) := \sum_{m=1}^{\infty} \frac{r_3(m^2)}{m^s} = 6(1 - 2^{1-s}) \frac{\zeta(s)\zeta(s-1)}{L(s,\chi_{-4})} \tag{2.13}$$

where

$$L(s,\chi_{-4}) := \sum_{m=1}^{\infty} (\frac{-4}{m}) m^{-s} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}. \tag{2.14}$$

The right hand side of (2.13) is a meromorphic function in the s-plane, which has a simple pole at $s = 2$ with residue

$$c_1 = \frac{3\zeta(2)}{L(2,\chi_{-4})} = \frac{\pi^2}{2L(2,\chi_{-4})},$$

and has no other poles for $\Re s > 1$. One could now proceed by a standard contour integral approach [3] to obtain $N_{\mathcal{L}}(\sqrt{2}T)$ equals $c_1 T^2$ plus a lower-order error term, so that $N_{\mathcal{L}}(T) = \frac{1}{2}c_1 T^2 + o(T^2)$. However we will derive (2.9) directly from the exact formula (2.12). We have,

$$\begin{aligned} N_{\mathcal{L}}(\sqrt{2}T) &= 1 + 2 \sum_{1 \le j \le log_2 T} \sum_{n=1}^{\lfloor \frac{T}{2^j} + \frac{1}{2} \rfloor} r_3((2n-1)^2) \\ &= 1 + 12 \sum_{1 \le j \le log_2 T} \sum_{n=1}^{\lfloor \frac{T}{2^j} + \frac{1}{2} \rfloor} (2n-1) \prod_{\substack{p \equiv 3 \pmod 4 \\ p \mid 2n-1}} (1 + \frac{2}{p} + ... + \frac{2}{p^{e_p(2n-1)}}). \end{aligned}$$

Expanding the products above and using $\sum_{n=1}^{U}(2n-1) = U^2 + O(U)$, one obtains

$$\begin{aligned} N_{\mathcal{L}}(\sqrt{2}T) &= 1 + 12 \sum_{k \ge 0} \sum_{P_k} \frac{2^k}{P_k} \left(\sum_{j \ge 1} \sum_{m=1}^{\lfloor \frac{T}{2^j P_k} + \frac{1}{2} \rfloor} (2m-1) P_k \right) \\ &= \frac{12}{4} \left(\sum_{k \ge 0} 2^k \sum_{\substack{j \ge 0, P_k \\ P_k < T/2^j}} \frac{1}{2^{2j} P_k^2} \right) T^2 + O\left((\sum_{k \ge 0} 2^k \sum_{P_k \le T} \frac{1}{P_k}) T \right), \qquad (2.15) \end{aligned}$$

in which $P_k = p_1^{e_1} ... p_k^{e_k}$ with all $p_i \equiv 3 \pmod 4$ and all $e_i \ge 1$. If the condition $P_k < T/2^j$ were dropped in the first sum in parentheses above, then it would sum to $\frac{\zeta(2)}{L(2, \chi_{-4})}$, as one sees by examining the associated Euler product, which is

$$\frac{\zeta(s)}{L(s, \chi_{-4})} = (1 - 2^{-s})^{-1} \prod_{p \equiv 3 \pmod 4} \frac{1 + p^{-s}}{1 - p^{-s}} = \sum_{m=1}^{\infty} \frac{a(m)}{m^s},$$

evaluated at $s = 2$, since $\frac{1+p^{-s}}{1-p^{-s}} = 1 + 2p^{-s} + 2p^{-2s} + ...$ The error introduced by truncating this Dirichlet series at $s = 2$ at $n \le T$ is bounded by

$$\sum_{m=T}^{\infty} \frac{a(m)}{m^2} \le \sum_{m=T}^{\infty} \frac{2^{\nu(m)}}{m^2} \le \sum_{m=T}^{\infty} \frac{d(m)}{m^2} = O(\frac{\log T}{T}),$$

in which $\nu(n)$ counts the number of distinct prime divisors of m (without multiplicity), $d(n)$ counts the number of divisors on n, and the last estimate uses [4] the fact ([20, Theorem 319]) that the average order of $d(n)$ is $\log n$. The remainder term in (2.15) is bounded by

[3] Integrate $F(s)$ against $\frac{T^s}{s} ds$ on a vertical contour $\Re s = c$ for some c with $1 < c < 2$, cf. Davenport [12, Chapter 17].

[4] Sum in blocks $2^j T \le n \le 2^{j+1} T$, viewing denominators as nearly constant and averaging over numerators.

$O(T(\log T)^2)$, because

$$\sum_{k\geq 0} 2^k \sum_{P_k \leq T} \frac{1}{P_k} \leq \sum_{m=1}^{T} \frac{2^{\nu(m)}}{m} \leq \sum_{m=1}^{T} \frac{d(m)}{m} \leq (\sum_{m=1}^{T} \frac{1}{m})^2 = O((\log T)^2).$$

Since $\zeta(2) = \frac{\pi^2}{6}$, these estimates combine to give

$$N_{\mathcal{L}}(\sqrt{2}T) = \frac{\pi^2}{2L(2,\chi_{-4})} T^2 + O(T\ (logT)^2).$$

Rescaling T yields (2.9).

To handle primitive Lorentz quadruples, we use the function $r_3^*(m)$ which counts the number of primitive integer representations of m as a sum of three squares, using positive, negative and zero integers. Then $r_3(m^2) = \sum_{d \mid m} r_3^*(d^2)$, which yields

$$F^*(s) := \sum_{m=1}^{\infty} \frac{r_3^*(m^2)}{m^s} = \frac{F(s)}{\zeta(s)} = 6(1 - 2^{1-s}) \frac{\zeta(s-1)}{L(s,\chi_{-4})}.$$

The function $F^*(s)$ is analytic in $\Re(s) > 1$ except for a simple pole at $s = 2$ with residue $\frac{6}{\pi^2} c_1$, which gives the constant in the main term of (2.10). To obtain the error estimate (2.10) one can use (2.9) and Möbius inversion, with $r_3^*(m^2) = \sum_{d \mid m} \mu(d) r_3((\frac{m}{d})^2)$. We omit details. $\square$

Remarks. (1) Various Dirichlet series associated to zero solutions of indefinite quadratic forms have meromorphic continuations to $\mathbb{C}$, cf. Andrianov [2]. These can used to obtain asymptotics for the number of solutions satisfying various side conditions.

(2) The real solutions of the homogeneous equation $Q_{\mathcal{L}}(w,x,y,z) = -w^2 + x^2 + y^2 + z^2 = 0$ form the *light cone* in special relativity.

3. Reduction Theory and Root Quadruples

In this section we describe, given an Apollonian circle packing and a Descartes quadruple in it, a reduction procedure which, if it halts, identifies within it a unique Descartes quadruple (a,b,c,d) which is "minimal". This quadruple is called the *root quadruple* of the packing. This procedure always halts for integral Apollonian packings.

Definition 3.1. The *Apollonian group* $\mathcal{A}$ is the group generated by the four integer 4×4

matrices

$$S_1 = \begin{bmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad S_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{bmatrix}$$

As mentioned earlier, the Apollonian group was introduced in the 1967 paper of Hirst[21], and was later used in Söderberg [46] and Aharonov and Stephenson [1] in studying Apollonian packings.

We view real Descartes quadruples $\mathbf{v} = (a,\ b,\ c,\ d)^T$ as column vectors, and the Apollonian group $\mathcal{A}$ acts by matrix multiplication, sending $\mathbf{v}$ to $M\mathbf{v}$, for $M \in \mathcal{A}$. The action takes Descartes quadruples to Descartes quadruples, because $\mathcal{A} \subset Aut_{\mathbb{Z}}(Q_{\mathcal{D}})$, the set of real automorphs of the Descartes integral quadratic form $Q_{\mathcal{D}}$ given in (2.3). That is, each such M satisfies

$$M^T Q_{\mathcal{D}} M = Q_{\mathcal{D}}, \qquad \text{for all } M \in \mathcal{A},$$

a relation which it suffices to check on the four generators $S_i \in \mathcal{A}$.

The elements S_j have a geometric meaning as corresponding to inversion in one of the four circles of a Descartes quadruple to give a new quadruple in the same circle packing, as explained in [17, Section 2]. That paper showed that this group with the given generators is a Coxeter group whose only relations are $S_1^2 = S_2^2 = S_3^2 = S_4^2 = I$.

The reduction procedure attempts to reduce the size of the elements in a Descartes quadruple by applying one of the generators S_j to take the quadruple $\mathbf{v} = (a,\ b,\ c,\ d)$ viewed as a column vector to the new quadruple $S_j\mathbf{v}$, until further decrease is not possible. Each packing has a well-defined *sign*, which is the sign of $L(\mathbf{v}) := a + b + c + d$ of any Descartes quadruple in the packing. (The sign is independent of the quadruple chosen in the packing; see Lemma 3.1(ii) below.) We describe the reduction procedure for packings with positive sign, those with $L(\mathbf{v}) := a + b + c + d > 0$; the reduction procedure for negative sign packings is obtained by conjugating by the inversion $(a, b, c, d) \to (-a, -b, -c, -d)$, which takes negative sign packings to positive sign packings. Thus we suppose $L(\mathbf{v}) := a+b+c+d > 0$ and for simplicity consider the case where the quadruple is ordered $a \le b \le c \le d$. We consider which $S_j\mathbf{v}$ can decrease the sum $L(\mathbf{v}) = a+b+c+d$, and show below this can never occur using S_1, S_2 or S_3. Note that $S_4(a,\ b,\ c,\ d)^T = (a,\ b,\ c,\ d')^T$ where $d' = 2(a+b+c) - d$.

Lemma 3.1. *Suppose that* $\mathbf{v} = (a,\ b,\ c,\ d)^T$ *is a real Descartes quadruple. Let* $\mathbf{v}$ *have its elements ordered* $a \le b \le c \le d$, *and set* $d' = 2(a+b+c) - d$.

(i) If $L(\mathbf{v}) = a+b+c+d > 0$, *then* $a+b \ge 0$, *with equality holding only if* $a = b = 0$ *and* $c = d$. *As a consequence, we always have* $b \ge 0$.

(ii) If $L(\mathbf{v}) = a+b+c+d > 0$, *then each* $L(S_j\mathbf{v}) > 0$. *In particular* $L(S_4\mathbf{v}) = a+b+c+d' > 0$.

(iii) If $a \ge 0$, *so that* $L(\mathbf{v}) = a+b+c+d \ge 0$, *then* $d' \le c \le d$. *If* $d' < c$ *then the matrix* S_4 *that changes* d *to* d' *strictly decreases the sum* $L(\mathbf{v}) := a+b+c+d$, *and it is the only generator of* $\mathcal{A}$ *that does so. If* $d' = c$ *then necessarily* $c = d = d'$ *and no generator* S_i *of* $\mathcal{A}$ *decreases* $L(\mathbf{v})$.

Proof. (i) If $a \ge 0$ then we are done, so assume $a < 0$. Suppose first that $0 \le b \le c \le d$. Let $x = -(a+b)$, $y = -ab$, so that $y \ge 0$. In terms of these variables, the Descartes equation becomes

$$2(x^2 + 2y + c^2 + d^2) = (-x + c + d)^2,$$

which simplifies to

$$(c-d)^2 + x^2 + 4y + 2(c+d)x = 0. \tag{3.1}$$

This equation cannot hold if $x > 0$, because all terms on the left are nonnegative, and some are positive, hence $a+b \ge 0$. If $x = 0$, then (3.1) implies $y = 0$ and $c = d$, whence $a = b = 0$ and $c = d$.

The remaining case is $a \le b < 0$. We first observe that in any Descartes quadruple, at least three terms have the same sign. Indeed the Descartes equation can be rearranged as

$$(a-b)^2 + (c-d)^2 = (a+b)(c+d),$$

and if $a \le b < 0 < c \le d$ then the left side is nonnegative while the right side is strictly negative, a contradiction. Thus we must have $a \le b \le c \le 0 < d$, since $a+b+c+d > 0$. Now the Descartes quadruple $(-d, -c, -b, -a)$ has $-d < 0 \le -c$, so the preceding case applies to show that $-d-c \geq 0$. We conclude that $a+b+c+d \leq 0$, contradicting the fact that $a+b+c+d > 0$. This completes the proof of (i).

(ii) The ordering of entries gives $L(S_1\mathbf{v}) \geq L(S_2\mathbf{v}) \geq L(S_3\mathbf{v}) \geq L(S_4\mathbf{v})$, so it suffices to prove $L(S_4\mathbf{v}) > 0$. The Descartes equation implies that

$$d, d' = a+b+c \pm 2q_{abc}, \qquad \text{where} \qquad q_{abc} = \sqrt{ab+bc+ca}.$$

We have $a+b+c+d' = 2(a+b+c) - 2\sqrt{ab+bc+ac} > 0$ because $a+b+c \geq 0$ (using (i)) and

$$(a+b+c)^2 - (ab+bc+ac) = \frac{1}{2}((a+b)^2 + (b+c)^2 + (a+c)^2) > 0.$$

(iii) The Descartes equation (1.1) gives

$$d' = a+b+c - 2\sqrt{ab+bc+ac}.$$

Thus

$$d' - c = a+b-2\sqrt{ab+bc+ac} \leq a+b - \sqrt{4(a+b)c} \leq a+b-\sqrt{(a+b)^2} = 0.$$

If $d' < c \leq d$ then the sum $L(\mathbf{v}') = a+b+c+d' < L(\mathbf{v})$, so the sum decreases. If S_i changes c to c', then $c' = 2(a+b+d) - c \geq 2(a+b+c) - c \geq c$ because $a+b \geq 0$ by (i), so the sum $L(\mathbf{v})$ does not decrease in this case. Similarly the sum does not decrease if S_i changes b to b' or a to a'. In the case of equality $d' = c$, one easily checks that $c = d = d'$, which forces $a = b = 0$, and no S_i decreases the sum $L(\mathbf{v})$. $\square$

Definition 3.2. A Descartes quadruple $\mathbf{v} = (a, b, c, d)$ with $L(\mathbf{v}) = a+b+c+d > 0$ is a *root quadruple* if $a \leq 0 \leq b \leq c \leq d$ and $a+b+c \geq d$.

Note that the last inequality above is equivalent to the condition $d' = 2(a+b+c) - d \geq d$.

Reduction algorithm.

Input: A real Descartes quadruple $(a,\ b,\ c,\ d)$ *with* $a+b+c+d > 0$.

(1) Test in order $1 \leq i \leq 4$ *whether some* S_i *decreases the sum* $a+b+c+d$. *It so, apply it to produce a new quadruple and continue.*

(2) If no S_i decreases the sum, order the elements of the quadruple in increasing order and halt.

The reduction algorithm takes real quadruples as input, and is not always guaranteed to halt. The following theorem shows that when the algorithm is given an integer Descartes quadruple as input, it always halts, and outputs a root quadruple. In the algorithm, the element S_i that decreases the sum necessarily decreases the largest element in the quadruple, leaving the other three elements unchanged. The proof below establishes that in all cases where a reduction is possible, the largest element of the quadruple is unique, so that the choice of S_i in the reduction step is unique. There do exist quadruples with a tie in the largest element, such as $(0,\ 0,\ 1,\ 1)$, but the vector $(a,\ b,\ c,\ d)$ cannot then be further reduced.

Theorem 3.2. (1) *If the reduction algorithm ever encounters some element $a < 0$, then it will halt at a root quadruple in finitely many more steps.*

(2) *If a, b, c, d are integers, then the reduction algorithm will halt at a root quadruple in finitely many steps.*

(3) *A root quadruple is unique if it exists. However an Apollonian circle packing may contain more than one Descartes configuration yielding this quadruple.*

Proof.

(1) Geometrically a Descartes quadruple with $a < 0$ describes a circle of of radius $1/a$ enclosing three mutually tangent circles of radii $1/b, 1/c, 1/d$. All circles in the packing lie inside this bounding circle of radius $1/a$. Each non-halting reduction produces a new circle of radius $1/d' > 1/d$, which covers an area of $\pi/{d'}^2$, and this is at least π/d^2. Since there is a total area of π/a^2 which can be covered, and all circles except the one with radius $1/a$ have disjoint interiors, this process must halt in at most $\left\lfloor (\frac{d}{a})^2 \right\rfloor$ steps.

(2) Let $q_{abc} = \sqrt{ab + bc + ac} = (a + b + c - d)/2 \in \mathbb{N}$. After each reduction, the sum $a + b + c + d$ decreases by $4q_{abc}$. By Lemma 3.1, the sum $a + b + c + d$ is bounded below by 0. Therefore this process halts after finitely many steps.

(3) If (a, b, c, d) is a root quadruple of an Apollonian packing, then the numbers a, b, c, d are the curvatures of the largest circles contained in this packing, hence they are unique. On

the other hand, the Apollonian packing may contain more than one copy of this quadruple, for example, $(-1, 2, 2, 3)$ appears twice in the packing shown in Figure 1, and $(0, 0, 1, 1)$ appears infinitely many times in the packing in Figure 1. $\Box$

Root quadruples lead to a partition of the set $Q(\mathbb{Z})$ of all integer Descartes quadruples. This set partitions into $Q(\mathbb{Z})^+ \cup \{(0,0,0,0)\} \cup Q(\mathbb{Z})^-$, where

$$Q(\mathbb{Z})^+ = \{(a,b,c,d) \in Q(\mathbb{Z}) : a+b+c+d > 0\} \tag{3.2}$$

and $Q(\mathbb{Z})^- = -Q(\mathbb{Z})^+$. Next we have the partition

$$Q(\mathbb{Z})^+ = \bigcup_{k=1}^{\infty} kQ(\mathbb{Z})^+_{prim} \ , \tag{3.3}$$

where $Q(\mathbb{Z})^+_{prim}$ enumerates all primitive integer Descartes quadruples in $Q(\mathbb{Z})^+$. These latter are exactly the Descartes quadruples occurring in all primitive integer Apollonian packings, so we may further partition $Q(\mathbb{Z})^+_{prim}$ into a union of the sets $Q(\mathcal{P}_{\mathcal{D}})$, where $Q(\mathcal{P}_{\mathcal{D}})$ denotes the set of all Descartes quadruples in the circle packing $\mathcal{P}_{\mathcal{D}}$ having primitive root quadruple $\mathcal{D}$, i.e.

$$Q(\mathbb{Z})^+_{prim} = \bigcup_{\substack{\text{primitive root} \\ \text{quadruple } \mathcal{D}}} Q(\mathcal{P}_{\mathcal{D}}). \tag{3.4}$$

We study the distribution of root quadruples in §4 and the set of integers in a given packing $\mathcal{P}_{\mathcal{D}}$ in §5 and §6.

By definition the Apollonian group labels all the (unordered) Descartes quadruples in a fixed Apollonian packing. We now show that it has the additional property that for a given integral Apollonian packing, the integer curvatures of all circles not in the root quadruple lie in one-to-one correspondence with the non-identity elements of the Apollonian group.

Theorem 3.3. *Let $\mathcal{P}_{\mathbf{v}}$ be the integer Apollonian circle packing with root quadruple $\mathbf{v} = (a,\ b,\ c,\ d)^T$. Then the set of integer curvatures occurring in $\mathcal{P}$, counted with multiplicity, consists of the four elements of $\mathbf{v}$ plus the largest elements of each vector $M\mathbf{v}$, where M runs over all elements of the Apollonian group $\mathcal{A}$.*

Proof. Let $M = S_{i_n} \cdots S_{i_1}$ be a reduced word in the generators of $\mathcal{A}$, that is $S_{i_k} \neq S_{i_{k+1}}$ for $1 \leq k < n$. The main point of the proof is that if $\mathbf{w}^{(n)} = S_{i_n} \cdots S_{i_1}\mathbf{v}$, then $\mathbf{w}^{(n)}$ is obtained from $\mathbf{w}^{(n-1)}$ by changing one entry, and the new entry inserted is always the largest entry in the new vector. (It may be tied for largest value.) We prove this by induction on n. In the base case $n = 1$, there are four possible vectors $S_i\mathbf{v}$, whose inserted entries are $a' = 2(b+c+d) - a$, $b' = 2(a+c+d) - b$, $c' = 2(a+b+d) - c$, and $d' = 2(a+b+c) - d$, respectively. Since $a \le b \le c \le d$ we have $d' \le c' \le b' \le a'$, and since $\mathbf{v}$ is a root quadruple, we have $d' \ge d$, as asserted.

For the induction step, where $n \ge 2$, there are only three choices for S_{i_n} since $S_{i_n} \ne S_{i_{n-1}}$. If the elements of $\mathbf{w}^{(n-1)}$ are labelled in increasing order as $w_1^{(n-1)} \le w_2^{(n-1)} \le w_3^{(n-1)} \le w_4^{(n-1)}$, then we may choose the labels (in case of a tie for the largest element) so that $w_4^{(n-1)}$ was produced at step $n-1$, by the induction hypothesis. Thus exchanging $w_4^{(n-1)}$ is forbidden at step n, hence if $w_4^{(n)}$ denotes the new value produced at the next step, then

$$\begin{aligned} w_4^{(n)} &\ge 2(w_1^{(n-1)} + w_2^{(n-1)} + w_4^{(n-1)} - w_3^{(n-1)} \\ &\ge 2w_1^{(n-1)} + 2w_2^{(n-1)} + w_4^{(n-1)} > w_4^{(n-1)}, \end{aligned} \tag{3.5}$$

because $w_1^{(n-1)} + w_2^{(n-1)} > 0$ by Lemma 3.1(i). This completes the induction step.

The inversion operation produces one new circle in the packing, namely the new value added in the Descartes quadruple, and (3.5) shows that its curvature is $||M\mathbf{v}||_\infty$, where $||\mathbf{w}||_\infty$ denotes the supremum norm of the vector $\mathbf{w}$.

Every circle in the packing is produced in this procedure, by definition of the Apollonian group. That all non-identity words $M \in \mathcal{A}$ label distinct circles is clear geometrically from the tree structure of the packing. $\square$

4. Distribution of Primitive Integer Root Quadruples

In this section we count integer Apollonian circle packings in terms of the size of their root quadruples. Recall that a Descartes quadruple $(a,\ b,\ c,\ d)$ is a *root quadruple* if $a \le 0 \le b \le c \le d$ and $d' = 2(a+b+c) - d \ge d > 0$. It suffices to study primitive packings, i.e. ones whose integer quadruples are relatively prime. We begin with a Diophantine characterization of root quadruples.

Theorem 4.1. *Given a solution* $(a, b, c, d) \in \mathbb{Z}^4$ *to the Descartes equation*

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2),$$

define (x, d_1, d_2, m) *by*

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ d_1 \\ d_2 \\ m \end{bmatrix} = \begin{bmatrix} x \\ d_1 - x \\ d_2 - x \\ -2m + d_1 + d_2 - x \end{bmatrix} . \tag{4.1}$$

Then $(x, d_1, d_2, m) \in \mathbb{Z}^4$ *satisfies*

$$x^2 + m^2 = d_1 d_2 \ . \tag{4.2}$$

Conversely, any solution $(x, d_1, d_2, m) \in \mathbb{Z}^4$ *to this equation yields an integer solution to the Descartes equation as above. In addition:*

(i) The solution (a, b, c, d) *is primitive if and only if* $\gcd(x, d_1, d_2) = 1$*.*

(ii) The solution (a, b, c, d) *with* $a + b + c + d > 0$ *is a root quadruple if and only if* $x < 0 \leq 2m \leq d_1 \leq d_2$*.*

Proof. The first part of the theorem requires, to have $m \in \mathbb{Z}$, that $a+b+c+d \equiv 0 \pmod 2$. This follows from the Descartes equation by reduction (mod 2).

For (i), note that $\gcd(x, d_1, d_2) = \gcd(a, b, c) = \gcd(a, b, c, d)$.

For (ii), the condition $a < 0 \leq b \leq c \leq d$ implies successively $x < 0$, $d_1 \leq d_2$, $d_1 - 2x = b - a \geq 0$, and $-2m + d_1 = d - c \geq 0$. Finally the root quadruple condition $d' = 2(a+b+c) - d \geq d \geq 0$ gives $d' = 2m \geq 0$. Thus $x < 0 \leq 2m \leq d_1 \leq d_2$. The converse implication follows similarly. $\square$

We now study primitive integer root quadruples with $a = -n$, for $n \in \mathbb{Z}_{\geq 0}$. Theorem 4.1 shows that they are in one-to-one correspondence with the integer solutions (m, d_1, d_2) to

$$n^2 + m^2 = d_1 d_2, \tag{4.3}$$

$$0 \leq 2m \leq d_1 \leq d_2 \quad \text{and} \quad \gcd(n, d_1, d_2) = 1 \ . \tag{4.4}$$

For each of $n = 0, 1, 2$, there is one primitive root quadruple with $a = -n$, namely, $(0, 0, 1, 1)$, $(-1, 2, 2, 3)$, $(-2, 3, 6, 7)$, respectively. For $n = 3$, there are two, $(-3, 4, 12, 13)$ and $(-3, 5, 8, 8)$. As an example of a nonsymmetric integral Apollonian circle packing, Figure 3 pictures the packing $(-6,\ 11,\ 14,\ 15)$.

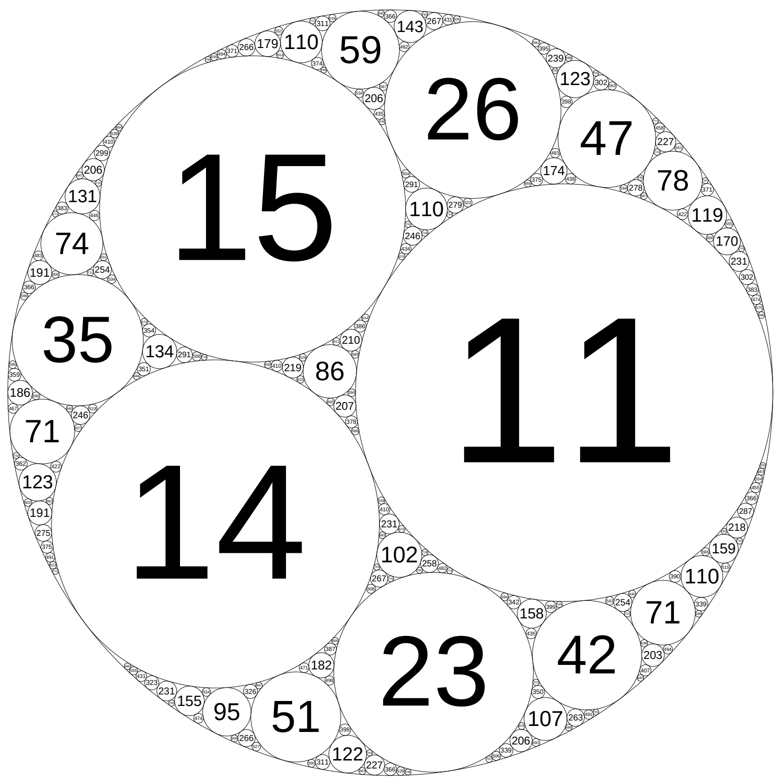

Figure 3: The nonsymmetric packing $(-6,\ 11,\ 14,\ 15)$.

Let $N^*_{root}(-n)$ denote the number of primitive root quadruples with negative element $-n$. One has $N^*_{root}(-n) \geq 1$ for all $n \geq 0$, since $(x,\ d_1,\ d_2,\ m) = (-n, 1, n^2, 0)$ in Theorem 4.1 produces the primitive root quadruple $(a,\ b,\ c,\ d) = (-n, n+1, n(n+1), n(n+1)+1)$. Table 1 and Table 2 present selected values of $N^*_{root}(-n)$.

n	$N(-n)$	n	$N(-n)$	n	$N(-n)$	n	$N(-n)$	n	$N(-n)$
1	1	11	4	21	10	31	9	41	11
2	1	12	6	22	7	32	9	42	18
3	2	13	4	23	7	33	14	43	12
4	2	14	5	24	10	34	9	44	14
5	2	15	6	25	6	35	10	45	14
6	3	16	5	26	7	36	14	46	13
7	3	17	5	27	10	37	10	47	13
8	3	18	7	28	10	38	11	48	18
9	4	19	6	29	8	39	14	49	15
10	3	20	6	30	10	40	10	50	11

Table 1: $N^*_{root}(-n)$ for small n

p	$N(-p)$	p	$N(-p)$	p	$N(-p)$	p	$N(-p)$
1009	253	3001	751	4007	1003	5011	1254
1013	254	3011	754	4013	1004	10007	2503
2003	502	4001	1001	5003	1252	10009	2503
2011	504	4003	1002	5009	1253	20011	5004

Table 2: $N^*_{root}(-p)$ for selected prime p.

Examination of numerical data led the the third author and S. Northshield independently to conjecture an exact formula for $N^*_{root}(-n)$, stated as Theorem 4.3 below. Northshield [37] obtained a proof, and we give another here. We establish this result by showing that $N^*_{root}(-n)$ can be interpreted as a class number, namely $N^*_{root}(-n) = h^{\pm}(-4n^2)$ where for $-4\Delta < 0$ the quantity $h^{\pm}(-4\Delta)$ counts the number of $GL(2,\mathbb{Z})$-equivalence classes of primitive integral binary quadratic forms

$$[A, B, C] := AT^2 + 2BTU + CU^2,$$

of discriminant $-4\Delta = 4B^2 - 4AC$; a form is primitive if $gcd(A, B, C) = 1$. By comparing $GL(2,\mathbb{Z})$-actions and $SL(2,\mathbb{Z})$-actions on positive definite binary quadratic forms one obtains the relation

$$h^{\pm}(-4\Delta) = \frac{1}{2}(h(-4\Delta) + a(-4\Delta)) \tag{4.5}$$

where $h(-4\Delta)$ is the usual binary quadratic form class number (for $SL(2,\mathbb{Z})$-equivalence) and $a(-4\Delta)$ counts the $SL(2,\mathbb{Z})$-equivalence classes containing an ambiguous reduced binary form of discriminant -4Δ (defined below). The exact formula in Theorem 4.3 is deduced from classical formulas for $h(-4n^2)$ and $a(-4n^2)$.

Theorem 4.2. *The Apollonian root quadruples $(-n, x, y, z)$ with $-n < 0 \le x \le y \le z$ are in one-to-one correspondence with positive definite integral binary quadratic forms of discriminant $-4n^2$ having nonnegative middle coefficient. The associated reduced binary quadratic form $[A, B, C] = AT^2 + 2BTU + CU^2$ is given by*

$$[A, B, C] := [-n + x,\ \frac{1}{2}(-n + x + y - z),\ \ -n + y].$$

Primitive root quadruples correspond to reduced binary quadratic forms having a nonnegative middle coefficient. In particular, the number $N^*_{root}(-n)$ of primitive root quadruples with least element $-n$ satisfiies

$$N^*_{root}(-n) = h^{\pm}(-4n^2),$$

where $h^{\pm}(-4n^2)$ is the number of $GL(2,\mathbb{Z})$-equivalence classes of positive definite primitive binary integral forms of discriminant $-4n^2$.

Remark. A positive definite form $[A, B, C]$ is *reduced* if $0 \le |2B| \le A \le C$. Reduced forms with nonnegative middle coefficients enumerate form clases under $GL(2,\mathbb{Z})$-action, rather than the $SL(2,\mathbb{Z})$-action studied by Gauss; the $GL(2,\mathbb{Z})$-action corresponds to the notion of equivalence of (definite) quadratic forms used by Legendre.

Proof. Recall that the Descartes quadratic form is

$$Q_{\mathcal{D}}(w, x, y, z) := (w + x + y + z)^2 - 2(w^2 + x^2 + y^2 + z^2).$$

A Descartes quadruple is any integer solution to $Q_{\mathcal{D}}(w,x,y,z)=0$ and a Descartes quadruple is *primitive* if $gcd(w,x,y,z)=1$. Any integer solution satisfies

$$w+x+y+z \equiv \ 0 \ (\mathrm{mod}\ 2),$$

as follows by reduction (mod 2). By definition a Descartes quadruple (w,x,y,z) satisfying $w+x+y+z>0$ is a *root quadruple* if and only if

(1) $w=-n \le 0 \le x \le y \le z$,

(2) $2(-n+x+y)-z \ge z$.

Condition (1) implies conversely that $w+x+y+z>0$, and condition (2) is equivalent to

$$-n+x+y-z \ge 0.$$

The integer solutions $Q_{\mathcal{D}}(w,x,y,z)=0$ with $w=-n<0$ are in one-to-one correspondence with integer representations of n^2 by the the ternary quadratic form

$$Q_T(X,Y,Z) := XY+XZ+YZ,$$

The correspondence between solutions to $Q_{\mathcal{D}}(w,x,y,z)=0$ and solutions to $Q_T(X,Y,Z)=n^2$ is given by

$$(X,Y,Z) := (\frac{1}{2}(w+x+y-z), \frac{1}{2}(w+x-y+z), \frac{1}{2}(w-x+y+z)).$$

The congruence condition $w+x+y+z \equiv 0 \ (\mathrm{mod}\ 2)$ on integer Descartes quadruples implies that (X,Y,Z) are all integers. The map is onto, because an explicit inverse map from an integer solution (X,Y,Z) is

$$(w,x,y,z) := (-n, n+X+Y, n+X+Z, n+Y+Z).$$

The primitivity condition $gcd(-n,x,y,z)=1$ on Descartes quadruples is equivalent to the primitivity condition $gcd(X,Y,Z)=1$. The "root quadruple conditions" (1) and (2) above are equivalent to the inequalities

$$0 \le X \le Y \le Z.$$

For any integer M, the integer solutions of $Q_T(X,Y,Z)=M$ are in one-to-one correspondence with integer representations of M by the determinant ternary quadratic form

$$Q_{\Delta}(A,B,C) := AC-B^2. \tag{4.6}$$

A solution (X, Y, Z) gives a solution to $Q_{\Delta}(A, B, C) = M$ under the substitution

$$(A, B, C) := (X + Y, X, X + Z).$$

The inverse map is

$$(X, Y, Z) := (B, A - B, C - B).$$

The "root quadruple conditions" above on (X, Y, Z) are easily checked to be equivalent to the inequalities

$$0 \leq 2B \leq A \leq C. \tag{4.7}$$

In this correspondence. B is necessarily an integer, so $2B$ is an even integer. The primitivity condition $gcd(X, Y, Z) = 1$ transforms to the primitivity condition $gcd(A, B, C) = 1$.

The conditions (4.7) give a complete set of equivalence classes of primitive positive definite integral binary forms of fixed discriminant $D = -4M$ under the action of $GL(2, \mathbb{Z})$. To show this, note that the conditions for the positive definite binary quadratic form $[A, B, C] := AT^2 + 2BTU + CU^2$ of discriminant $D = 4B^2 - 4AC = -4M$ to be a reduced form in the $SL(2, \mathbb{Z})$ sense are

$$0 \leq |2B| \leq A \leq C. \tag{4.8}$$

It is known that all positive definite integral forms are $SL(2, \mathbb{Z})$-equivalent to a reduced form, and that all reduced forms are $SL(2, \mathbb{Z})$-inequivalent, except for $[A, B, A] \approx [A, -B, A]$ and $[2A, A, C] \approx [2A, -A, C]$, where $\approx$ denotes $SL(2, \mathbb{Z})$-equivalence. Since $GL(2, \mathbb{Z})$ adds only the action of $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, it follows that all are equivalent to a form with $B \geq 0$, and all of these are $GL(2, \mathbb{Z})$- inequivalent.

Combining these two steps, where one takes $M = n^2$ in the second step, associates to any Descartes quadruple $(-n, x, y, z)$ the definite binary quadratic form

$$[A, B, C] := [-n + x, \frac{1}{2}(-n + x + y - z), -n + y] \tag{4.9}$$

of discriminant $D := (2B)^2 - 4AC = -4n^2$. Furthermore this form is reduced in the $GL(2, \mathbb{Z})$-sense (4.7) if and only if $(-n, x, y, z)$ is a root quadruple. The primitivity condition $gcd(-n, x, y, z) = 1$ transforms to $gcd(A, B, C) = 1$; conversely, $gcd(A, B, C) = 1$ implies $gcd(-n, x, y, z) = 1$. $\square$

Using Dirichlet's class number formula we obtain an explicit formula for $N^*_{root}(-n)$.

Theorem 4.3. *For $n > 1$ the number $N^*_{root}(-n)$ of primitive integral root quadruples satisfies*

$$N^*_{root}(-n) = \frac{n}{4}\prod_{p|n}(1 - \frac{\chi_{-4}(p)}{p}) + 2^{\omega(n)-\delta_n-1}, \tag{4.10}$$

where $\chi_{-4}(n) = (-1)^{(n-1)/2}$ for n and 0 for even n, $\omega(n)$ denotes the number of distinct primes dividing n, and $\delta_n = 1$ if $n \equiv 2 (\mathrm{mod}\ 4)$ and $\delta_n = 0$ otherwise.

Proof. The condition for a positive definite form $[A, B, C]$ to be $SL(2,\mathbb{Z})$-reduced is that $0 \leq 2|B| \leq A \leq C$. The form classes under $GL(2,\mathbb{Z})$-equivalence are derived from the $SL(2,\mathbb{Z})$-equivalence classes using the coset decomposition

$$GL(2,\mathbb{Z}) = \{I, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\} SL(2,\mathbb{Z}).$$

The action of

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

maps the set of $SL(2,\mathbb{Z})$-reduced forms into itself, hence

$$h^{\pm}(-4n^2) = \frac{1}{2}(h(-4m^2) + a(-4m^2)) \tag{4.11}$$

where $a(-4m^2)$ denotes the number of primitive reduced ambiguous forms of discriminant $-4m^2$, where we define a *reduced ambiguous form* to be any $SL(2,\mathbb{Z})$-reduced form of the shape $[A, 0, C]$, $[B, 2B, C]$, and $[A, 2B, A]$, with $B > 0$. (These are exactly the reduced forms with $B \geq 0$ and $[A, B, C] \approx [A, -B, C]$, see Mathews [30, pp. 69–72].) A formula for the general class number $h(d)$ was given by Dirichlet, and we use the version in Landau [25]. For a discriminant $d < 0$, Landau [25, Theorem 209], states that

$$h(d) = \frac{w_d}{2\pi}\sqrt{|d|}L(1, \chi_d)$$

in which $L(s, \chi_d) = \sum_{n=1}^{\infty} \chi_d(n) n^{-s}$, with $\chi_d(n) = (\frac{d}{n})$ a real Dirichlet character (mod $|d|$), and $w_d = 2$ if $d < -4$, with $w_{-4} = 4$. Thus for $d = -4n^2$ with $n > 1$,

$$h(-4n^2) = \frac{2n}{\pi} L(1, \chi_{-4n^2}).$$

Landau [25, Theorems 214] also gives

$$L(1, \chi_{-4n^2}) = \prod_{p|n}(1 - \frac{\chi_{-4}(p)}{p})\ L(1, \chi_{-4}).$$

Combining this with Lambert's formula $L(1, \chi_{-4}) = \frac{\pi}{4}$, we obtain, $h(-4) = 1$ and, for $n > 1$, the class number formula

$$h(-4n^2) = \frac{n}{2} \prod_{p|n} (1 - \frac{\chi_{-4}(p)}{p}). \tag{4.12}$$

It remains to determine $a(-4n^2)$. We claim that, for $n > 1$,

$$a(-4n^2) = \begin{cases} 2^{\omega(n)-1} & \text{if } \ n \equiv 2 \pmod 4, \\ 2^{\omega(n)} & \text{otherwise.} \end{cases} \tag{4.13}$$

To prove this we consider for each $j \geq 0$ separately the cases $n = 2^j s$, with s odd. There are three types of ambiguous reduced forms to consider:

(1) $[A, 0, C]$ with $0 < A \leq C$;

(2) $[2A, A, C]$ with $0 < 2A \leq C$;

(3) $[A, B, A]$, with $0 < 2B \leq A$.

The primitivity requirement is that $gcd(A, B, C) = 1$. For type (1), the discriminant condition gives $n^2 = AC$. Since $gcd(A, C) = 1$ each relatively prime factorization $n = pq$ with $gcd(p, q) = 1$ gives a candidate pair $(A, C) = (p^2, q^2)$, and the requirement $A \leq C$ rules out half of these, provided $n > 1$, for then $A = C$ cannot occur. We conclude that there are exactly $2^{\omega(n)-1}$ solutions of this type, when $n > 1$. The case $n = 1$ is exceptional, with $A = C = 1$ as the only solution.

For the remaining count we must group types (2) and (3) together. For type (2) we have $-4n^2 = 4A^2 - 8AC$ which gives $n^2 = A(2C - A)$. The condition $gcd(A, C) = 1$ implies $gcd(A, 2C - A) = 1$ or 2. For type (3) we have $-4n^2 = 4B^2 - 4A^2$ which gives $n^2 = (A + B)(A - B)$. The condition $gcd(A, B) = 1$ implies $gcd(A + B, A - B) = 1$ or 2.

We first suppose $j = 1$. This is the easiest case because there are no solutions of types (2) and (3). Indeed, for type (2) we have $A(2C - A)$ is even, so A must be even, and $2C - A$ is even. But $gcd(A, C) = 1$ means C is odd, so A and $2C - A$ are not congruent (mod 4). Thus one of them is divisible by 4, hence 8 divides $A(2C - A) = n^2$ which contradicts n^2 being divisible by 4 but not 8. For type (3) we have $(A+B)(A-B)$ is even, so both factors are even. At least one of A, B is odd since $gcd(A, B) = 1$, so the two factors are incongruent (mod 4), hence one of them is divisible by 4, so again we deduce that 8 divides n^2, a contradiction.

We next suppose $j = 0$. Then n is odd and the primitivity requirement for type (2) forms becomes $gcd(A, 2C-A) = 1$, and for type (3) forms becomes $gcd(A+B, A-B) = 1$. Consider a factorization $n^2 = p^2q^2$ with $gcd(p,q) = 1$. For type (2), setting this equal to the factorization $A(2C - A)$ yields $A = p^2, C = 1/2(p^2 + q^2)$. The condition $2A \leq C$ becomes

$$p^2 \leq \frac{1}{3}q^2.$$

For type (3), setting $n = q^2p^2$ equal to the factorization $(A+B)(A-B)$ yields $A = 1/2(q^2 + p^2),\ B = 1/2(q^2 - p^2)$. The condition $B \geq 0$ gives $p^2 \leq q^2$ and the condition $2B \leq A$ gives $3p^2 \geq q^2$; thus $q^2/3 \leq p^2 \leq q^2$, and we have $q^2/3 \neq p^2$ by the relative primality condition. These two cases both require $p^2 \leq q^2$, and they have disjoint ranges for p^2 and cover the whole range $0 < p^2 \leq q^2$. Thus exactly half the factorizations lead to one solution and the other half to no solution, and we obtain exactly $2^{\omega(n)-1}$ solutions of types (2) and (3).

Finally we suppose $j \geq 2$. In these cases the primitivity requirement for type (2) forms becomes $gcd(A, 2C - A) = 2$, and for type (3) forms becomes $gcd(A + B, A - B) = 2$. We have $n^2 = 2^{2j}p^2q^2$ and there are several splittings of factors to consider, for example $A = 2^{2j-1}p^2, 2C - A = 2q^2$ or $A = 2q^2, 2C - A = 2^{2j-1}p^2$ in type (2) and $A + B = 2^{2j-1}q^2, A - B = 2p^2$ or vice versa in case (3). In all cases, a careful counting of factorizations finds exactly half the possible factorizations lead to solutions of types (2) and (3), giving $2^{\omega(n)-1}$ solutions of types (2) and (3). We omit the details.

Adding up these cases gives $2^{\omega(n)}$ solutions in total when $j = 0$ or $j \geq 2$ and $2^{\omega(n)-1}$ solutions if $j = 1$. This proves the claim.

Combining (4.11) and (4.12) with the claim yields the desired result. $\square$

The exact formula for $N^*_{root}(-n)$ for prime $n = p$ gives $N^*_{root}(-p) = \frac{p+3}{4}$ if $p \equiv 1 (\mathrm{mod}\ 4)$ and $N^*_{root}(-p) = \frac{p+5}{4}$ if $p \equiv 3 (\mathrm{mod}\ 4)$, as in Table 2. The exact formula also yields the following asymptotic upper and lower bounds for $N^*_{root}(-n)$.

Theorem 4.4. *There are positive constants C_1 and C_2 such that, for all $n \geq 3$,*

$$C_1 \frac{n}{\log\log n} < N^*_{root}(-n) < C_2\ n \log\log n. \tag{4.14}$$

Proof. We use the exact formula (4.10) of Theorem 4.3. The second term $\frac{1}{2}a(-4n^2)$ grows like $O(n^\epsilon)$ and is negligible compared to the first term. To estimate the first term we use inequalities of the form

$$\prod_{p|n}(1-\frac{1}{p}) \geq c_1 \frac{1}{\log\log n}. \tag{4.15}$$

and

$$\prod_{p|n}(1+\frac{1}{p}) \leq c_2 \log\log n. \tag{4.16}$$

To establish (4.16), we note that its left side is upper bounded by the product taken over the first k distinct primes, with k chosen minimal so that the product exceeds n. Thus

$$\log(\prod_{p|n}(1+\frac{1}{p})) \leq \sum_{p \leq \log n/\log\log n} \log(1+\frac{1}{p}) \leq \sum_{p \leq \log n/\log\log n} \frac{1}{p} \leq \log\log\log n$$

for large enough n, using [20, Theorem 427], and exponentiating this yields (4.16). The inequality (4.15) follows from this because $\prod_p(1-\frac{1}{p^2})$ converges. $\square$

The bounds of Theorem 4.4 are sharp up to a multiplicative constant. We can take n to be a product of all small primes $p \equiv 3 \pmod 4$ to achieve a sequence of values with $N^*_{root}(-n) >> n \log\log n$, and to be a product of all small primes $p \equiv 1 \pmod 4$ to achieve a sequence of values with $N^*_{root}(-n) << \frac{n}{\log\log n}$.

Remark. Theorem 4.2 gives a one-to-one correspondence between Descartes quadruples containing $-n$ and integral binary quadratic forms of discriminant $-4n^2$ having nonnegative middle coefficient. These Descartes configurations are exactly the ones that contain the outer circle of the packing as one of their circles. Under this correspondence the reduction algorithm of §3 for Descartes quadruples matches the Gaussian reduction algorithm for positive definite binary quadratic forms.

A related correspondence holds for the exceptional integral Apollonian packing $(0,0,1,1)$ having $n = 0$. There is a unique primitive reduced positive definite form of determinant 0, namely $Q = [0,0,1] = U^2$, which is an ambiguous form. In this case the special Descartes quadruples are those that contain one of the two infinite edges of the packing as one of their circles. They are essentially Descartes configurations of “Ford circles”. and on these configurations the reduction algorithm corresponds to an additive variant of the continued fraction algorithm. For futher information see Ford [15] or Rademacher [38, pp. 41–46], [39, pp. 264–267].

5. Integers Represented by a Packing: Asymptotics

In this section we study the ensemble of integer curvatures that occur in an integer Apollonian circle packing, where integers are counted with the multiplicity that they occur in the packing. Their asymptotics are known to be related to the Hausdorff dimension of the residual set of the packing, as follows from work of Boyd described in Theorem 5.2 below. At the end of the section we begin the study of the set of integer curvatures that occur, counted without multiplicity.

The *residual set* of a disk packing $\mathcal{P}$ (not necessarily an Apollonian packing) is the set remaining after all the (open) disks in the packing are removed, including any disks with "center at infinity." For a general disk packing $\mathcal{P}$, we denote the Hausdorff dimension of the residual set by $\alpha(\mathcal{P})$ and call it the *residual set dimension* of the packing. The definition of Hausdorff dimension can be found in Falconer [14], who also studies the residual sets of Apollonian packings in [14, pp. 125–131.].

The residual sets of Apollonian packings all have the same Hausdorff dimension, which we denote by α. This is a consequence of the equivalence of such residual sets under Möbius transformations (see [17, Sect. 2]), using also the fact that the Hausdorff dimension strictly exceeds one, as follows from results described below.

The *exponent* or *packing constant* $e(\mathcal{P})$ of a bounded circle packing $\mathcal{P}$ (not necessarily an Apollonian packing) is defined to be

$$e(\mathcal{P}) := \sup\{e : \sum_{C \in \mathcal{P}} r(C)^e = \infty\} = \inf\{e : \sum_{C \in \mathcal{P}} r(C)^e < \infty\},$$

in which $r(C)$ denotes the radius of the circle C. This number has been extensively studied in the literature, beginning in 1966 with the work of Melzak [33, Theorem 3], who showed that in any circle packing that covers all but a set of measure zero one has $\sum_{C \in \mathcal{P}} r(C) = \infty$. He constructed a circle packing with $e(\mathcal{P}) = 2$ and showed for Apollonian packings that $e(\mathcal{P})$ lies strictly between 1.035 and 1.99971. He conjectured that the minimal value of $e(\mathcal{P})$ is attained by an Apollonian circle packing. In 1967 J. Wilker [51] showed that all osculatory circle packings $\mathcal{P}$, which include all Apollonian circle packings, have the same exponent $e(\mathcal{P})$, which we call the *osculatory packing exponent* e. He also showed that $e \geq 1.059$. Later Boyd [3], [4], [7] improved this to $1.300 < e < 1.314$. Recent non-rigorous computations of Thomas and Dhar [48] estimate the Apollonian packing exponent to be 1.30568673 with a possible error of 1 in the last digit.

The relation between the packing exponent and the residual set dimension of Apollonian packings was resolved by an elegant result of D. Boyd [6].

Theorem 5.1. (Boyd) *The exponent e of any bounded Apollonian circle packing is equal to the Hausdorff dimension α of the residual set of any Apollonian circle packing.*

The inequality $e \geq \alpha$ follows from a 1966 result of Larman [27], and in 1973 Boyd proved the matching upper bound $\alpha \geq e$. A simpler proof of the upper bound was later given by C. Tricot [49].

Given a bounded circle packing $\mathcal{P}$ we define the *circle-counting function* $N_{\mathcal{P}}(T)$ to count the number of circles in the packing whose radius of curvature is no larger than T, i.e., whose radius is at least $\frac{1}{T}$. Boyd [7] proved the following improvement of the result above.

Theorem 5.2. (Boyd) *For a bounded Apollonian circle packing $\mathcal{P}$, the circle-counting function $N_{\mathcal{P}}(T)$ satisfies*

$$\lim_{T\to\infty} \frac{log\ N_{\mathcal{P}}(T)}{log\ T} = \alpha, \tag{5.1}$$

where α is the Hausdorff dimension of the residual set. That is, $N_{\mathcal{P}}(T) = T^{\alpha+o(1)}$ as $T \to \infty$.

. Theorem 3.3 showed that the curvatures of all circles in the packing, excluding the root quadruple, can be enumerated by the elements of the Apollonian group $\mathcal{A}$. ¿From this one can derive a relation between the number of elements of $\mathcal{A}$ having height below a given bound T and the Hausdorff dimension α. We measure the *height* of an element $M \in \mathcal{A}$ using the Frobenius norm

$$||M||_F := (tr[M^T M])^{1/2} = (\sum_{i,j} M_{ij}^2)^{1/2}. \tag{5.2}$$

Theorem 5.3. *The number of elements $N_T(\mathcal{A})$ of height at most T in the Apollonian group $\mathcal{A}$ satisfies*

$$N_T(\mathcal{A}) = T^{\alpha+o(1)}, \tag{5.3}$$

as $T \to \infty$, where α is the Hausdorff dimension of the residual set of any Apollonian packing.

In order to prove this result, we establish two preliminary lemmas.

Lemma 5.4. *Let $M = S_{i_m} \cdots S_{i_2} S_{i_1} \in \mathcal{A}$, the Apollonian group, and suppose that $i_j \neq i_{j+1}$ for $1 \leq j \leq m-1$, and $m \geq 2$. In each row k of M,*

(i) $M_{kl} \leq 0$ *if* $l = i_1$,

(ii) $M_{kj} \geq |M_{kl}|$ *for* $l = i_1$ *and* $j \neq l$.

Proof. The lemma follows by induction on m. It is true for $m = 1$, since each matrix S_i has i^{th} column negative (or zero).

Suppose (i)–(ii) hold for $M' = S_{i_m} \cdots S_{i_2}$. Suppose, for convenience, that $i_1 = 1$. Then

$$M = M' S_{i_1} = \begin{bmatrix} -M'_{11} & 2M'_{11} + M'_{12} & 2M'_{11} + M'_{13} & 2M'_{11} + M'_{14} \\ -M'_{21} & 2M'_{21} + M'_{22} & 2M'_{21} + M'_{23} & 2M'_{21} + M'_{24} \\ -M'_{31} & 2M'_{31} + M'_{32} & 2M'_{31} + M'_{33} & 2M'_{31} + M'_{34} \\ -M'_{41} & 2M'_{41} + M'_{42} & 2M'_{41} + M'_{43} & 2M'_{41} + M'_{44} \end{bmatrix}.$$

Since $i_2 \neq i_1 = 1$ all $M'_{i1} \geq 0$ by (ii) of the induction hypothesis, so $M_{i1} = M'_{i1} \leq 0$ gives (i). Next, note that

$$M_{kj} = 2M'_{k1} + M'_{kj} \geq 2M'_{k1} - |M'_{kl}| \geq M'_{k1} = |M_{k1}|$$

since $M'_{kj} \geq |M'_{kj}|$ and $M'_{kj} \geq -|M'_{kl}|$ in all cases by (ii). This completes the induction step in this case. The arguments when $i_1 = 2, 3$, or 4 are similar. $\square$

Lemma 5.5. *Let $\mathbf{v} = (a, b, c, d)^T$ be an integer root quadruple with $a < 0$. Then there are positive constants $c_0 = c_0(\mathbf{v})$ and $c_1 = c_1(\mathbf{v})$ depending on $\mathbf{v}$ such that*

$$c_0 \|M\|_F \leq ||M\mathbf{v}||_\infty \leq c_1 \|M\|_F, \quad \textit{for all} \quad M \in \mathcal{A}\ . \tag{5.4}$$

Proof. For the upper bound, we have

$$||M\mathbf{v}||_\infty \leq 2|M\mathbf{v}| \leq 2\|M\|_F |\mathbf{v}|\ , \tag{5.5}$$

so we may take $c_1 = 2|\mathbf{v}|$.

For the lower bound, we first show that if $M = S_{i_m} \cdots S_{i_2} S_{i_1}$ with $i_j \neq i_{j+1}$ and $i_1 = 1$, we have

$$||M\mathbf{v}||_\infty \geq \frac{1}{2} \|M\|_F\ . \tag{5.6}$$

The vector $\mathbf{v}$ has sign pattern $(-,+,+,+)$ and Lemma 5.4 shows that M has first column nonpositive elements and other columns nonnegative. Thus all terms in the product $M\mathbf{v}$ are nonnegative, and hence

$$(M\mathbf{v})_i \geq \sum_{j=1}^{4} |M_{ij}||\mathbf{v}_j| \geq \sum_{j=1}^{4} |M_{ij}| \ ,$$

because $a < 0$ implies $\min(|a|,|b|,|c|,|d|) \geq 1$. Thus

$$||M\mathbf{v}||_\infty \geq \frac{1}{4}\sum_{i,j} |M_{ij}| \geq \frac{1}{2}\|M\|_F \ .$$

It remains to deal with the cases where $i_1 = 2,\ 3$ or 4. By Theorem 3.3, the value $||M\mathbf{v}||_\infty$ gives the curvature of a particular circle in the packing, and this circle lies in one of the four lunes pictured in Figure 3 according to the value of i_m.

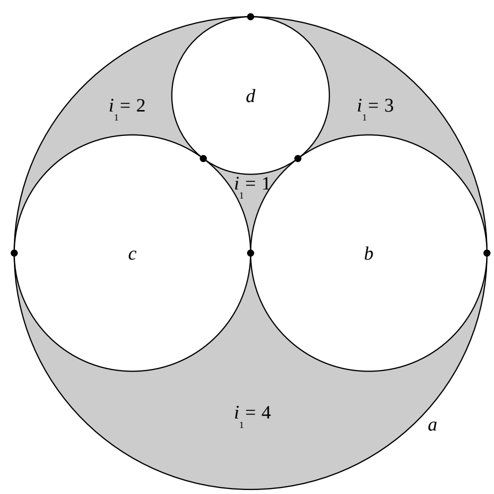

Figure 4: Four lunes of Descartes quadruple.

The bound (5.6) applies to all circles in the central lune corresponding to $i_m = 1$. For the remaining cases, we use the fact that there exists a Möbius transformation $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with $\phi \in Aut(\mathcal{P})$, which fixes the Descartes configuration corresponding to $\mathbf{v}$ but cyclically permutes the four circles $a \to b \to c \to d$. In particular ϕ also cyclically permutes the four lunes $i_1 = 1 \to i_1 = 2 \to i_1 = 3 \to i_1 = 4$. Now ϕ maps the center of circle d to the center of circle a, which is the point at infinity, and maps the point at infinity to the center of circle b. It follows that the stretching factor of the map ϕ inside the four lunes is bounded above and below by positive absolute constants c_2 and c_2^{-1}. Since ϕ maps the lune $i_1 = 4$ to $i_1 = 1$ we conclude for cases where $i_1 = 4$ that

$$||M\mathbf{v}||_\infty \geq \frac{1}{2c_2}\|M\|_F \ . \tag{5.7}$$

Applying the same argument to ϕ^2 and ϕ^3 gives the similar bound for the cases $i_1 = 3$ and $i_1 = 2$. We conclude that the lower bound in (5.4) holds with $c_0 = \frac{1}{2c_2}$. $\square$

Proof of Theorem 5.3. Pick a fixed quadruple having $a < 0$, say $\mathbf{v} = (-1, 2, 2, 3)$, and let $\mathcal{P}_{\mathbf{v}}$ be the associated Apollonian packing. By Theorem 3.3, each $M \in \mathcal{A}$ corresponds to a circle of curvature $||M\mathbf{v}||_\infty$ in $\mathcal{P}_{\mathbf{v}}$, and all circles are so labelled except the four circles in $\mathbf{v}$. Lemma 5.5 shows that each $\|M\|_F < T$ produces a circle of curvature at most c_1T. Now Theorem 5.2 asserts there are at most $T^{\alpha+o(1)}$ such circles, hence $N_T(\mathcal{A}) \leq T^{\alpha+o(1)}$. Conversely Lemma 5.5 implies that each circle of curvature $||M\mathbf{v}||_\infty \leq T$ comes from a matrix $M \in \mathcal{A}$ with $\|M\|_F \leq \frac{1}{c_0}T$. Since there are at least $T^{\alpha+o(1)}$ such circles, we obtain $N_T(\mathcal{A}) \geq T^{\alpha+o(1)}$, as desired. $\square$

Can the estimate of Theorem 5.3 be sharpened to obtain an asymptotic formula? A. Gamburd has pointed out to us that the method of Lax and Phillips [28] might prove useful in studying this question.

We now turn to a different question: How many different integers occur, counted without multiplicity, in a given integral Apollonian circle packing $\mathcal{P}_{\mathbf{v}}$? This seems to be a difficult problem. It is easy to prove that at least $cT^{1/2}$ of all integers less than T occur in a given packing. This comes from considering the largest elements of the vectors $\{(S_1S_2)^j\mathbf{v} : j = 1, 2, ...\}$, where $\mathbf{v}$ is a root quadruple, which are curvatures in the packing, by Theorem 3.3 above. These values grow like j^2 (see the example (1) in §7). Concerning the true answer to the question above, we propose the following conjecture.

Positive Density Conjecture. *Each integral Apollonian packing represents a positive fraction of all integers.*

Theorem 5.2 shows that the average number of representations of an integer n grows like $n^{\alpha-1}$, which goes rapidly to infinity as $n \to \infty$. Therefore one might guess that all sufficiently large integers are represented. However in the next section we will show there are always some congruence restrictions on which integers occur. There we formulate a stronger version of this conjecture and present numerical evidence concerning it.

6. Integers Represented by a Packing: Congruence Conditions

In this section we study congruence restrictions on the set of integer curvatures which occur in a primitive integral Apollonian packing.

We first show that there are always congruence restrictions (mod 12).

Theorem 6.1. *In any primitive integral Apollonian packing, the Descartes quadruples (mod 12) all fall in exactly one of four possible orbits. The first orbit Y (mod 12) consists of all permutations of*

$$\{(0,0,1,1),\ (0,1,1,4),(0,1,4,9),(1,4,4,9),(4,4,9,9)\}\ (\text{mod } 12). \tag{6.1}$$

The other three orbits are $(3,3,3,3)-Y$*,* $(6,6,6,6)+Y$ *and* $(9,9,9,9)-Y$ *(mod 12).*

Remark. Each orbit contains only 4 different residue classes (mod 12), hence 8 residue classes (mod 12) are excluded as curvature values.

Proof. A straightforward computation, using the action of the Apollonian group (mod 12), shows that (up to ordering) the set of all quadruples without common factors of 2 or 3 (mod 12) consists of the list below, which are grouped into eight orbits under the action of the Apollonian group (mod 12).

$$\begin{array}{llclllll}
(1) & Y & = & (0,0,1,1) & (0,1,1,4) & (0,1,4,9) & (1,4,4,9) & (4,4,9,9); \\
(2) & (3,3,3,3)-Y & = & (6,6,11,11) & (2,6,11,11) & (2,3,6,11) & (2,2,3,11) & (2,2,3,3); \\
(3) & (6,6,6,6)+Y & = & (3,3,10,10) & (3,6,7,10) & (3,7,10,10) & (6,7,7,10) & (6,6,7,7); \\
(4) & (9,9,9,9)-Y & = & (0,0,5,5) & (0,5,5,8) & (0,5,8,9) & (5,8,8,9) & (8,8,9,9); \\
(5) & -Y & = & (0,0,11,11) & (0,8,11,11) & (0,3,8,11) & (3,8,8,11) & (3,3,8,8); \\
(6) & (3,3,3,3)+Y & = & (0,0,7,7) & (0,4,7,7) & (0,3,4,7) & (3,4,4,7) & (3,3,4,4); \\
(7) & (6,6,6,6)-Y & = & (2,2,9,9) & (2,2,5,9) & (2,5,6,9) & (2,5,5,6) & (5,5,6,6); \\
(8) & (9,9,9,9)+Y & = & (9,9,10,10) & (1,9,10,10) & (1,6,9,10) & (1,1,6,10) & (1,1,6,6);
\end{array}$$

To check the orbit structure is as given, note that the action of the four generators of the Apollonian group on the five elements of the orbit Y is summarized in the following transition matrix:

$$\frac{1}{4}\begin{pmatrix} 2 & 2 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 2 \end{pmatrix}.$$

We may view this matrix as the transition matrix of a Markov chain (after rescaling each row to be stochastic), and find that the action is transitive and the stationary distribution is $(\frac{1}{10}, \frac{1}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{10})$. The other seven orbits have the same transition matrix and the same stationary distribution as Y.

There exist integral solutions to the Descartes equation in all of the congruence classes (mod 12) in the list above. However we recall that a Descartes quadruple (a, b, c, d) coming from an Apollonian packing satisfies the extra condition

$$a + b + c + d > 0. \tag{6.2}$$

In the rest of the proof we show that this extra condition excludes half of the orbits above, namely orbits (5)- (8).

As a preliminary, we observe that any integer solution (a, b, c, d) to the Descartes equation (1.1) yields a unique integer solution to the equation

$$4m^2 + 4a^2 + n^2 = l^2, \tag{6.3}$$

and vice-versa. Here the solution to (6.3) is given by

$$\begin{bmatrix} a \\ n \\ l \\ m \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 1 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b - c \\ 2a + b + c \\ \frac{1}{2}(d - a - b - c) \end{bmatrix}. \tag{6.4}$$

In the reverse direction, an integer solution to (6.3) gives one to the Descartes equation via

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ n \\ l \\ m \end{bmatrix} = \begin{bmatrix} a \\ \frac{1}{2}(l - 2a + n) \\ \frac{1}{2}(l - 2a - n) \\ 2m + l - a \end{bmatrix}. \tag{6.5}$$

Solutions to the Descartes equation satisfy a congruence (mod 2) which guarantee that the maps above take integral solutions to integral solutions, in both directions. Now (6.3) gives

$$l^2 \geq 4a^2 + m^2 \geq 2(|a| + |m|)^2 \geq (|a| + |m|)^2, \tag{6.6}$$

and equality holds if and only if $\ell = a = m = 0$. In particular, if $\ell > 0$, then (6.6) gives

$$\ell > |a| + |m|. \tag{6.7}$$

We assert that any integer solution (a, b, c, d) to the Descartes equation has

$$a + b + c + d > 0 \qquad \text{if and only if} \quad l > 0. \tag{6.8}$$

To prove this, note that if $a + b + c + d > 0$ then by Lemma 3.1 (i) we have

$$\ell = 2a + b + c = (a + b) + (a + c) \geq 0.$$

Equality can hold here only if $a = b = c = 0$, which implies $d = 0$, which contradicts the assumption $a + b + c + d > 0$. Conversely, if $\ell > 0$, then, using (6.7),

$$a + b + c + d = 2\ell + 2m - 2a \geq 2(\ell - |a| - |m|) > 0,$$

so (6.8) is proved.

Claim : No primitive integer Descartes quadruples with $a + b + c + d > 0$ occur in the orbits (5)–(8).

We prove the claim for orbit (8); the arguments to rule out orbits (5), (6), (7) are similar. We argue by contradiction. Suppose there were such a solution in orbit (8). Since the Apollonian group acts transitively on the orbit, and preserves the condition $a + b + c + d > 0$, there would be such a quadruple $(a, b, c, d) \equiv (1, 1, 6, 6) (\text{mod } 12)$. In this case $l = 2a + b + c \equiv 9 (\text{mod } 12)$ and

$$m \equiv \frac{1}{2}(6 - 6 + 1 + 1) \equiv 1 (\text{mod } 6),$$

which gives $m^2 \equiv 1 (\text{mod } 12)$. Now (6.3) gives

$$(l + 2m)(l - 2m) = l^2 - 4m^2 = 4a^2 + n^2 > 0. \tag{6.9}$$

Since $a + b + c + d > 0$ we have $l > 0$ by (6.8). Then in the equation above at least one of the factors on the left side must be positive, hence they both are. Consider $l + 2m > 0$. We have

$$l + 2m \equiv 9 \pm 2 (\text{mod } 12) \equiv 3 (\text{mod} 4). \tag{6.10}$$

Consider any prime $p \equiv 3 (\text{mod } 4)$ dividing $l + 2m$. Then it divides $4a^2 + n^2$, which it must divide to an even power, say p^{2e}, with $a \equiv n \equiv 0 (\text{mod } p^e)$. If p also divides $l - 2m$, then it would divide both l and m, and then (6.5) would imply that it divides $\gcd(a,\ b,\ c,\ d)$, which contradicts the primitivity assumption $\gcd(a,\ b,\ c,\ d) = 1$. Therefore p does not divide

$l-2m$, and we conclude from (6.9) that $p^{2e}||l+2m$. It follows that all primes $p \equiv 3 (\text{mod } 4)$ that divide $l+2m$ do so to an even power, hence we must have have $l+2m \equiv 1 (\text{mod } 4)$, a contradiction. This rules out orbit (8), which proves the claim in this case.

Theorem 6.1 follows from the claim. □

At the end of this section we present numerical evidence that suggests that these congruences (mod 12) are the only congruence restrictions for the integer packing $(-1,2,2,3)$. However there are stronger modular restrictions (mod 24) that apply to other integer packings. For example, in the packing $(0,0,1,1)$ (Fig. 2), any curvature which occurs must be congruent to $0,1,4,9,12$ or 16 (mod 24) (these are the quadratic residues modulo 24). Thus only 6 classes (mod 24) can occur rather than the 8 classes allowed by Theorem 6.1.

It seems likely that the full set of congruence restrictions possible (mod m) is attained for m a small fixed power $2^a 3^b$, perhaps even $m = 24$. We are a long way from proving this. As evidence in its favor, we prove the following result, which shows that all residue classes modulo m do occur for any m relatively prime to 30.

Theorem 6.2. *Let $\mathcal{P}$ be a primitive integral Apollonian circle packing. For any integer m with $gcd(m,30)=1$, every residue class modulo m occurs as the value of some circle curvature in the packing $\mathcal{P}$.*

Proof. Observe that the s-term product $W(s) = \dots S_2 S_1 S_2 S_1$ is

$$\begin{pmatrix} -s & s+1 & s(s+1) & s(s+1) \\ -(s-1) & s & s(s-1) & s(s-1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(where the top two rows are interchanged if s is even). Of course, the two non-trivial rows can be placed anywhere by choosing the two matrices from the set S_1, S_2, S_3, S_4 appropriately.

If $(a,b,c,d)^T$ is a quadruple in $\mathcal{P}$ then the product

$$W(s)(a,b,c,d)^T =$$

$$(-sa+(s+1)b+s(s+1)c+s(s+1)d, -(s-1)a+sb+s(s-1)c+s(s-1)d, c, d)^T \quad (6.11)$$

is also in $\mathcal{P}$ as well. Let $\mathcal{J}$ denote the set of all rows $(\alpha,\beta,\gamma,\delta)$ which can occur in a product of matrices taken from S_1, S_2, S_3, S_4. Thus, if $(\alpha,\beta,\gamma,\delta) \in \mathcal{J}$ then so are:

$(-\alpha, 2\alpha+\beta, 2\alpha+\gamma, 2\alpha+\delta)$, $(2\beta+\alpha, -\beta, 2\beta+\gamma, 2\beta+\delta)$, $(2\gamma+\alpha, 2\gamma+\beta, -\gamma, 2\gamma+\delta)$, and $(2\delta+\alpha, 2\delta+\beta, 2\delta+\gamma, -\delta)$. Therefore, $(-\alpha, 2\alpha+\beta, 2\alpha+\gamma, 2\alpha+\delta), (3\alpha+2\beta, -2\alpha-\beta, 6\alpha+2\beta+\gamma, 6\alpha+2\beta+\delta)$, $(-3\alpha-2\beta, 4\alpha+3\beta, 12\alpha+6\beta+\gamma, 12\alpha+6\beta+\delta), \ldots$, and in general,

$$(-r\alpha-(r-1)\beta, (r+1)\alpha+r\beta, r(r+1)\alpha+r(r-1)\beta+\gamma, r(r+1)\alpha+r(r-1)\beta+\delta) \tag{6.12}$$

are all in $\mathcal{J}$ for all r (as well as all permutations of these). Now substitute $(\alpha,\beta,\gamma,\delta) = (s(s+1), s(s+1), -s, s+1) \in \mathcal{J}$ into (6.12). This shows that the row vector

$$\rho := (-(2r-1)s(s+1), (2r+1)s(s+1), 2r^2s(s+1)-s, 2r^2s(s+1)+s+1) \in \mathcal{J}. \tag{6.13}$$

The sum of the last two coordinates of ρ is

$$4r^2s(s+1)+1 = r^2((2s+1)^2-1)+1 = r^2(x^2-1)+1 = u^2-r^2+1 \tag{6.14}$$

where $u = rx$ and $x = 2s+1$. The g.c.d. of these two summands must divide their difference, which is $2s+1$. It is well known (and easy to show) that for any prime power p^w with $p > 5$, in at least one of the pairs $\{1,2\}, \{4,5\}$ and $\{9,10\}$ are both nonzero quadratic residues modulo p^w. For each $p \mid m$, let $\{a_p, a_p+1\}$ denote such a pair. Define u_p and r_p so that

$$u_p^2 \equiv a_p, \qquad r_p^2 \equiv a_p+1 \pmod{p^w} \tag{6.15}$$

where p^w is the largest power of p dividing m. Since $\gcd(r_p, p) = 1$ then we can define $x_p \equiv u_p r_p^{-1} \pmod{p^w}$. We can guarantee that x_p is odd by adding a multiple of p_w if necessary. Hence, for these choices, the expression in (6.14) is 0 modulo p^w, i.e.,

$$r_p^2(x_p^2-1)+1 \equiv 0 \pmod{p^w}. \tag{6.16}$$

Of course, we can use the values $r_p + kp^w$ and $x_p + lp^w$ in place of r_p and x_p in (6.16) for any k and l. Note that $\gcd(x_p, p) = 1$. Letting p range over all prime divisors of m, then by the Chinese Remainder Theorem, there exist X (odd) and R such that

$$R^2(X^2-1)+1 \equiv 0 \pmod{p^w} \tag{6.17}$$

for all $p^w| \ m$. Thus,

$$R^2(X^2-1)+1 \equiv 0 \pmod{m}, \qquad gcd(X, m) = 1. \tag{6.18}$$

Hence, by (6.13) and (6.14) we can find a row modulo m in $\mathcal{J}$ of the form $(C, D, A, -A) \pmod m$ where it easy to check that $\gcd(A, m) = 1$.

We can now apply the transformation preceding (6.12) to $(C, D, A, -A)$ to get the following rows modulo m in $\mathcal{J}$:

$$\begin{array}{rrrl} (C, & D, & A, -A) & \pmod m \\ (2A + C, & 2A + D, & -A, A) & \pmod m \\ (4A + C, & 4A + D, & A, -A) & \pmod m \\ (6A + C, & 6A + D, & -A, A) & \pmod m \\ & \cdots & & \end{array}$$

and more generally

$$(4tA + C, 4tA + D, A, -A) \pmod m \in \mathcal{J} \quad \text{for all } t \geq 0. \tag{6.19}$$

Suppose for the moment (and we will prove this shortly) that we can find $(a, b, c, d)^T \in \mathcal{P}$ with $\gcd(a + b, m) = 1$. Taking the inner product of the row in (6.19) with $(a, b, c, d)^T$, we get the curvature value

$$(4tA + C, 4tA + D, A, -A) \cdot (a, b, c, d)^T \pmod m \tag{6.20}$$

$$\equiv 4A(a + b)t + Ca + Db + Ac - Ad \pmod m. \tag{6.21}$$

Since $\gcd(4A(a + b), m) = 1$ then these values range over a complete residue system modulo m as t runs over all positive integers. The proof will be complete now if we can establish the following result.

Claim. If $\mathcal{P}$ is a primitive packing then for any odd $m \geq 1$, there exists $(a, b, c, d)^T \in \mathcal{P}$ with $\gcd(a + b, m) = 1$.

Proof of Claim. First recall that for any $(a, b, c, d) \in \mathcal{P}$, we have $\gcd(a, b, c) = 1$. We have also seen by (6.11), if $(a, b, c, d) \in \mathcal{P}$ then for any $r > 1$,

$$(A(r), B(r), C(r), D(r)) :=$$

$$(-ra + (r + 1)b + r(r + 1)(c + d), -(r - 1)a + rb + r(r - 1)(c + d), c, d) \in \mathcal{P} \tag{6.22}$$

as well. Define $q(r)$ to be the sum of the first two components of this vector:

$$q(r) := A(r) + B(r) = 2(c + d)r^2 - 2(a - b)r + a + b.$$

Let p denote a fixed odd prime. We show that

$$q(r) \not\equiv 0 \pmod{p} \quad \text{for some} \quad r \geq 1. \tag{6.23}$$

Suppose to the contrary that $q(r) \equiv 0 \pmod{p}$ for all r. Thus,

$$\begin{aligned} q(0) &\equiv a + b \equiv 0 \pmod{p} \\ q(1) &\equiv 2(c+d) - 2(a-b) + (a+b) \equiv 2(c+d) - a + 3b \equiv 0 \pmod{p} \\ q(2) &\equiv 8(c+d) - 4(a-b) + (a+b) \equiv 8(c+d) - 3a + 5b \equiv 0 \pmod{p} \end{aligned}$$

which implies $a \equiv b \equiv c + d \equiv 0 \pmod{p}$. However, since $a = b + c + d \pm 2\sqrt{b(c+d) + c\, d}$ then $cd \equiv 0 \pmod{p}$, i.e., $c \equiv 0$ or $d \equiv 0 \pmod{p}$. This would imply that $\mathcal{P}$ is not primitive, a contradiction which establishes (6.23).

To finish proving the claim, for each $p|m$ let r_p satisfy $q(r_p) \not\equiv 0 \pmod{p}$. Then we have

$$q(r_p + kp) \equiv q(r_p) \not\equiv 0 \pmod{p}$$

for all $k \geq 0$. By the Chinese Remainder Theorem one can find R and S such that $q(R+kS) \not\equiv 0 \pmod{p}$ for all $p \mid m$ and all k. In particular, $\gcd(q(R), m) = \gcd(A(R) + B(R), m) = 1$, and the Claim is proved. $\square$

Which integers occur as curvatures, when the congruence conditions are taken into account? We consider numerical data for two cases. The first case is the packing with root quadruple $(-1, 2, 2, 3)$, where Theorem 6.1 permits only values $2, 3, 6$, or $11 \pmod{12}$. Not all such integers appear in the Apollonian packing $(-1, 2, 2, 3)$, for example in the class $6 \pmod{12}$ the value 78 is missed. In Table 3 we present the missing values in these residue classes for the first million integers. Only 61 integers congruent to $2, 3$, or 6 do not occur in the packing $(-1, 2, 2, 3)$, the largest being 97287 (see Table 3), and no integers $11 \pmod{12}$ are missed. This data suggests that there are finitely many missing values in total, with 97287 being the largest one.

Our second example is the packing with root quadruple $(0, 0, 1, 1)$. As mentioned above, there are congruence conditions (mod 24) in this case. Table 4 presents numerical data on exceptional values for the allowed congruence classes (mod 24) up to $T = 10^7$. There is a

$n \equiv 3 \pmod{12}$

159	207	243	435	603	711	1923	2175	2319	3711
4167	4959	4995	5283	6015	6879	7863	10095	10923	11295
12063	16311	16515	18051	19815	21135	23175	28323	41655	48075
68055	97287								

$n \equiv 6 \pmod{12}$

78	246	342	834	1422	2010	2022	2454	2718	2766
3150	3402	3510	3774	4854	6018	6666	7470	10638	12534
13154	13206	20406	24270	32670	42186	45258	55878		

$n \equiv 2 \pmod{12}$

13154

Table 3: Missing integers in the packing $(-1, 2, 2, 3)$ up to 10^6

much larger set of exceptional values, and it appears more equivocal whether the full list of exceptional values is finite. However we think it is.

The numerical examples above support the idea that for any fixed integer Apollonian packing and for sufficiently large integers a finite list of congruence conditions will be the only obstruction to existence. We therefore propose the following strengthening of the Density Conjecture.

Strong Density Conjecture. *In any primitive integral Apollonian packing, all sufficiently large integers occur, provided they are not excluded by congruence conditions.*

In further support of the Strong Density Conjecture, we note an analogy to a number-theoretic conjecture of Zaremba [53], who conjectured that there exists an absolute constant b (possibly $b = 5$) such that each sufficiently large positive integer can be represented by some continuant with digits bounded above by b. In other words, given any integer $m > 1$, there exists an integer $a < m$ (a relatively prime to m) such that the simple continued fraction $[0, c_1, \cdots, c_r] = a/m$ has partial denominators $c_i \leq b$. Fix b, and let M be the set of all pairs (a, m) with the above property. There is a linear recurrence for the pairs (a, m) which is similar to that of the Descartes quadruples, since if the terms in the continued fraction of a/m are bounded by b, then so are those for the fractions $1/(i + a/m)$, $i = 1, 2, \ldots, b$. Zaremba's conjecture is saying that all the integers m will appear in some pair of M. This conjecture is

$n \equiv 0 \pmod{24}$

48	120	360	528	552	720	888	912	1080
1176	1272	1392	1560	1704	1848	1968	2184	2208
2736	2880	3240	3408	3552	4080	4392	4464	4584
4680	4896	5040	5088	5760	6192	6888	7272	8280
8880	9792	10680	10920	10944	11760	11928	13152	14160
14328	16008	17160	17232	17520	18000	19320	20712	23160
25896	26472	26760	27552	27600	27768	29424	29688	30288
31440	34440	34488	35232	36408	36648	36816	37968	38928
39168	43056	43392	45240	46056	50448	52800	58728	59400
66120	74976	80280	82200	87192	93216	96912	96960	107016
108240	117480	121680	133392	137280	138360	165360	201480	399000
424560	496080							

$n \equiv 12 \pmod{24}$

132	252	300	468	636	780	1140	1476	1572
1980	2100	2148	2628	2820	2868	3012	3492	3828
3900	4212	4692	5028	5148	5340	5796	6516	6684
6900	7380	7908	8772	10020	10212	10260	10380	10548
11268	11868	12876	13572	14100	14244	14724	14916	15300
15588	19260	19620	20940	21732	22908	23652	24252	24804
25140	25812	26100	26124	27660	28860	29532	30540	31092
31932	36564	37908	38772	39780	41460	41964	44988	46980
52260	52788	61596	67308	69324	69420	75900	76908	79740
88140	101940	120300	135252	185580	188748	220308	228780	234660
354540	422820	472548	926820	1199820				

$n \equiv 1, 4$ or $9 \pmod{24}$

241	340	748	2980	5452	11380	45652	16617	21825

$n \equiv 16 \pmod{24}$

208	328	712	1168	2488	3400	5200	13600	15088	116896

Table 4: Missing integers in the packing $(0, 0, 1, 1)$, up to 10^7

currently still open. But as in the Apollonian packing, consideration of the Hausdorff dimension of the set $E_b = \{a/m : (a,m) \in M\}$ is suggestive. Namely, let $S_b(m)$ be the number of a's such that $(a,m) \in M$. If $S_b(m) \sim m^\beta$, then $\sum m^\beta m^{-x}$ converges iff $x \geq \beta + 1$. Since the abscissa of convergence of the series $\sum S_b(m) m^{-x}$ is equal to twice the Hausdorff dimension γ of E_b (see T. Cusick [10]), then $\beta = 2\gamma - 1 \approx .0624 > 0$. Thus the "expected" number of appearances of m in the pairs of M is $m^\beta \gg 1$.

7. The Growth of Descartes Quadruples in a Packing

The circles in an integral Apollonian circle packing, starting from the root quadruple, are enumerated by the elements of the Apollonian group. The graph of this group is a rooted infinite tree with four edges meeting each vertex, with each vertex labelled by a nontrivial word in the generators of the Apollonian group. (Such a word satisfies the condition that any two adjacent generators in the word are unequal.) Starting from the root node, there are 4 nodes at depth 1, and at each subsequent level there are three choices of generators at each node, so there are $4 \times 3^{n-1}$ words of length n labelling depth n circles. How are the curvatures of the circles at depth n distributed? We consider the maximum value, the minimum value, and the median value. In the process we also determine the joint spectral radius of the generators of the Apollonian group.

We begin with the maximum value. We define for $n = 4m + i$ with $0 \leq i \leq 3$, the reduced word T_n of length n given by

$$T_n := T_i (S_4 S_3 S_2 S_1)^m, \tag{7.1}$$

with $T_i = I, S_1, S_2 S_1, S_3 S_2 S_1$ for $0 \leq i \leq 3$, respectively.

Theorem 7.1. *Let $\mathbf{v} = (a,b,c,d)$ be any root quadruple with $a \leq b \leq c \leq d$ and $a < 0$, $a+b+c+d > 0$. Then for any reduced word W of length n in the generators $\{S_1, S_2, S_3, S_4\}$ of the Apollonian group,*

$$||W\mathbf{v}||_\infty \leq ||T_n \mathbf{v}||_\infty \ . \tag{7.2}$$

Proof. Write $W = S_{i_n} S_{i_{n-1}} \cdots S_{i_1}$ and set $\mathbf{w}^{(n)} = W\mathbf{v}$ and $\mathbf{v}^{(n)} = T_n \mathbf{v}$. Write the elements of $\mathbf{w}^{(n)}$ and $\mathbf{v}^{(n)}$ in increasing order as

$$w_1^{(n)} \leq w_2^{(n)} \leq w_3^{(n)} \leq w_4^{(n)} \quad \text{and} \quad v_1^{(n)} \leq v_2^{(n)} \leq v_3^{(n)} \leq v_4^{(n)} \ .$$

The idea of the proof is that T_n always inverts with respect to the circle of smallest curvature, and in fact produces the largest curvature vector in a strong lexicographic sense. More precisely, we prove by induction on $n \geq 1$ that

$$w_i^{(n)} \leq v_i^{(n)} \quad \text{for} \quad 1 \leq i \leq 4 \tag{7.3}$$

and

$$w_4^{(n)} - w_1^{(n)} \leq v_4^{(n)} - v_1^{(n)} \ . \tag{7.4}$$

For the base case $n = 1$, we have $\mathbf{v}^{(1)} = (a', b, c, d)$ where $a' = 2(b + c + d) - a = ||S_1\mathbf{v}||_\infty$. If $b' = 2(a + c + d) - b = ||S_2\mathbf{v}||_\infty$ and $c' = ||S_3\mathbf{v}||_\infty$, $d' = ||S_4\mathbf{v}||_\infty$ then $a \leq b \leq c \leq d$ gives $d' \leq c' \leq b' \leq a'$, and (7.3) holds for $n = 1$, since $d' \geq d$ because $\mathbf{v}$ is a root quadruple.

For the induction step, a reduced word has $i_n \neq i_{n-1}$. The forbidden move $S_{i_{n-1}}$ is the one that replaces $w_4^{(n-1)}$ with $2(w_1^{(n-1)} + w_2^{(n-1)} + w_3^{(n-1)}) - w_4^{(n-1)}$. Now the induction hypothesis gives

$$\begin{aligned}
w_1^{(n)} &\leq w_2^{(n-1)} \leq v_2^{(n-1)} = v_1^{(n)} \\
w_2^{(n)} &\leq w_3^{(n-1)} \leq v_3^{(n-1)} = v_2^{(n)} \\
w_3^{(n)} &\leq w_4^{(n-1)} \leq v_4^{(n-1)} = v_3^{(n)}
\end{aligned}$$

and

$$\begin{aligned}
w_4^{(n)} &\leq 2(w_2^{(n-1)} + w_3^{(n-1)} + w_4^{(n-1)}) - w_1^{(n-1)} \\
&\leq 2(w_2^{(n-1)} + w_3^{(n-1)}) + w_4^{(n-1)} + (w_4^{(n-1)} - w_1^{(n-1)}) \\
&\leq 2(v_2^{(n-1)} + v_3^{(n-1)}) + v_4^{(n-1)} + (v_4^{(n-1)} - v_1^{(n-1)}) \\
&= v_4^{(n)} \ .
\end{aligned}$$

For the remaining inequality, suppose first that $w_1^{(n)} = w_2^{(n-1)}$. Then

$$\begin{aligned}
w_4^{(n)} - w_1^{(n)} &= [2(w_2^{(n-1)} + w_3^{(n-1)} + w_4^{(n-1)}) - w_1^{(n-1)}] - w_2^{(n-1)} \\
&= w_2^{(n-1)} + 2w_3^{(n-1)} + w_4^{(n-1)} + (w_4^{(n-1)} - w_1^{(n-1)}) \\
&\leq v_2^{(n-1)} + 2v_3^{(n-1)} + v_4^{(n-1)} + (v_4^{(n-1)} - v_1^{(n-1)}) \\
&= v_4^{(n)} - v_1^{(n)} \ .
\end{aligned}$$

If, however, $w_1^{(n)} = w_1^{(n-1)}$, then

$$\begin{aligned} w_4^{(n)} - w_1^{(n)} &\leq [2(w_1^{(n-1)} + w_3^{(n-1)} + w_4^{(n-1)}) - w_2^{(n-1)}] - w_1^{(n-1)} \\ &\leq 2(w_2^{(n-1)} + w_3^{(n-1)} + w_4^{(n-1)}) - w_1^{(n-1)} - w_2^{(n-1)} \\ &\leq v_4^{(n)} - v_1^{(n)} , \end{aligned}$$

using the previous inequality. This completes the induction step. $\square$

The maximum growth rate of the elements at level n of the Apollonian group is also describable in terms of the joint spectral radius of the generators $\{S_1, S_2, S_3, S_4\}$ of the Apollonian group.

Definition 7.1. Given a finite set of $n \times n$ matrices $\Sigma = \{M_1, \ldots, M_s\}$ the *joint spectral radius* $\sigma(\Sigma)$ is

$$\sigma(\Sigma) := \limsup_{k\to\infty} \left\{ \max_{1\leq i_1,\cdots,i_k\leq s} \sigma(M_{i_1}\cdots M_{i_k})^{1/k} \right\} ,$$

where $\sigma(M) := \max\{|\lambda| : \lambda \text{ eigenvalue of } M\}$ is the spectral radius of M.

The notion of joint spectral radius has appeared in many contexts, including wavelets and fractals; see Daubechies and Lagarias [11] for a discussion and references. In general it is hard to compute, but here we can obtain an explicit answer.

Theorem 7.2. *The joint spectral radius for the generators $\Sigma = \{S_1, S_2, S_3, S_4\}$ of the Apollonian group is $\sigma(\Sigma) = \theta^{1/4}$ where*

$$\theta = \frac{1}{2}\left(1 + \sqrt{5} + \sqrt{2 + 2\sqrt{5}}\right) \approx 2.890 . \tag{7.5}$$

It is attained by $M = S_4S_3S_2S_1$.

Proof. Pick a fixed root quadruple with $a < 0$, say $\mathbf{v} = (-1, 2, 2, 3)$, and consider the associated packing $\mathcal{P}_{\mathbf{v}}$. Lemma 5.5 asserts that

$$c_0\|M\|_F \leq ||M\mathbf{v}||_\infty \leq c_1\|M\|_F, \quad \text{all} \quad M \in \mathcal{A} . \tag{7.6}$$

We use the well-known fact that, for any real $n \times n$ matrix M,

$$\sigma(M) = \lim_{k\to\infty} \|M^k\|_F^{1/k} . \tag{7.7}$$

Now (7.6) gives for any reduced word $M = S_{i_s} \cdots S_{i_2} S_{i_1} \in \mathcal{A}$ with $i_k \neq i_{k-1}$ that

$$\sigma(M)^{1/s} = \lim_{k\to\infty} (||M^k \mathbf{v}||_\infty)^{\frac{1}{ks}} ,$$

Choosing $k = 4n$, Theorem 7.1 yields

$$\sigma(M)^{1/s} \leq \lim_{n\to\infty} ||T_{4ns}\mathbf{v}||_\infty^{\frac{1}{4ns}} .$$

Since $T_{4ns} = (S_4S_3S_2S_1)^{ns}$, this gives

$$\begin{aligned} \sigma(M) &\leq \lim_{n\to\infty} ||(S_4S_3S_2S_1)^{ns}\mathbf{v}||_\infty^{\frac{1}{4ns}} \\ &\leq \sigma(S_4S_3S_2S_1)^{1/4} . \end{aligned}$$

Choosing $M = S_4S_3S_2S_1 \in \mathcal{A}$ attains equality (with $s = 4$), which determines the joint spectral radius. A computation reveals that the characteristic polynomial of $M = S_4S_3S_2S_1$ is $X^4 - 2X^3 - 2X^2 - 2X + 1 = 0$ which factors as

$$(X^2 + (-1 + \sqrt{5})X + 1)(X^2 - (1 + \sqrt{5})X + 1) = 0 .$$

Its spectral radius is given by (7.5). □

The minimal growth rate of any reduced word of length $2n$ is attained by the word $W_{2n} = (S_4S_3)^n$. If $\mathbf{v} = (a, b, c, d)$ is a root quadruple with $a < 0$, then the supremum norm

$$||W_{2n}\mathbf{v}||_\infty = n(n + 1)(a + b) - nc + (n - 1)d \tag{7.8}$$

grows quadratically with n. We omit the easy proof of this fact.

To conclude this section, we consider the "average value" of $||W\mathbf{v}||_\infty$ over all reduced words W in $\mathcal{A}$ of length n, which we define to be the *median* of this distribution. (The elements of the distribution are exponentially large, so the median is a more appropriate quantity to consider than the mean value.) Let T_n denote the median. We expect that its growth rate should be related to the Hausdorff dimension α of the limit set of the Apollonian packing. The results of §5 lead to the heuristic that one should expect

$$\lim_{n\to\infty} \frac{1}{n} \log T_n = \frac{\log 3}{\alpha} . \tag{7.9}$$

We leave the proof (or disproof) of this as an open problem.

8. Open questions

There remain many open questions concerning integral Apollonian circle packings. We list a few of these here.

(1) In any primitive integral Apollonian packing $\mathcal{P}$, just four distinct residue classes modulo 12 can occur as curvature values in $\mathcal{P}$. For example, for $\mathcal{P} = (0,0,1,1)$, these values are $0,1,4,9 \pmod{12}$ while for $\mathcal{P} = (-1,2,2,3)$, they are $2,3,6,11 \pmod{12}$. As we noted in Section 6, it seems likely that in the packing $(-1,2,2,3)$, all sufficiently large integers congruent to $2,3,6$ and $11 \pmod{12}$ actually do occur. However, in the packing $(0,0,1,1)$, instead of the 8 residue classes $0,1,4,9,12,13,16,22 \pmod{24}$ which we might expect to occur, the classes 13 and 22 are completely missing. Again, computation suggests that only finitely many values in the other 6 are missing in $(0,0,1,1)$. Is it true that in any integral Apollonian packing, the only congruence restrictions on the curvature values are for the modulus 24? However, in no case can we even show that the set of values which do occur has positive upper density.

(2) Is there a direct way for determining the root quadruple to which a given Descartes quadruple belongs? The only way we currently know uses the reduction algorithm described in Section 3.

(3) Concerning root quadruples, what are the asymptotics of the total number of root quadruples having Euclidean height below T?

(4) We have not proved any reasonable lower bound on the number of integers below T that occur as curvatures in a fixed integral Apollonian packing. For how large a β can one prove asymptotically that at least $T^{\beta+o(1)}$ integers occur in every such packing?

(5) All of the preceding questions can also be raised for integral Apollonian packing of spheres in 3 dimensions, as discussed in [19]. For example, what are the modular restrictions (if any) for the Descartes quintuples (a,b,c,d,e) occurring in the packing with root quintuple $(0,0,1,1,1)$?

(6) There exist strongly integral Apollonian packings, in which the circles all have integer curvatures and also the curvature$\times$centers of the circles are Gaussian integers, where the circle centers are coordinatized as complex numbers. (See [18].) Most questions investigated for integral packings can also be asked for strongly integral packings. If we write (x, xX) for a circle with curvature x and (complex) center X, then the pairs (x, xX) must also satisfy various

modular constraints. For example, modulo 12, the standard integral packing (i.e., starting with the circles $(-1,0),(2,1),(2,-1)$ has just 20 types of circles, namely,

$$\begin{aligned}(x, xX) \quad = \quad & (2,1),(2,3),(2,5),(2,7),(2,9),(2,11)\\ & (3,2i),(3,4i),(3,8i),(3,10i)\\ & (6,3+4i),(6,3+8i),(6,9+4i),(6,9+8i)\\ & (11,0),(11,4),(11,8),(11,6i),(11,4+6i),(11,8+6i)\end{aligned}$$

and there are just 120 different four-circle configurations. What are the asymptotics of these types and configurations? What is the characterization of the integral (complex) vectors (x, xX) that can appear in a given packing?

We hope to return to some of these issues in a future paper.

Acknowledgments. The authors wish to acknowledge the insightful comments of Arthur Baragar, William Duke, Andrew Odlyzko, Eric Rains, and Neil Sloane at various stages of this work. We also thank the referee for many useful comments and historical references.

References

[1] D. Aharonov and K. Stephenson, Geometric sequences of discs in the Apollonian packing, Algebra i Analiz **9** (1997), No. 3, 104–140. [English version: St. Petersburg Math. J. **9** (1998), 509–545.]

[2] A. N. Andrianov, Dirichlet series that correspond to representations of zero by indefinite quadratic forms, Algebra i Analiz **1** (1989), No. 3, 71–82. [English Version: St. Petersburg Math. J. **1** (1990), 635–646.]

[3] D. Boyd, The disk-packing constant, Aequationes Math. **7** (1971), 182–193.

[4] D. Boyd, Improved bounds for the disk-packing constant, Aequationes Math. **9** (1973), 99–106.

[5] D. Boyd, The osculatory packing of a three-dimensional sphere, Canadian J. Math. **25** (1973), 303–322.

[6] D. Boyd, The residual set dimension of the Apollonian packing, Mathematika **20** (1973), 170–174.

[7] D. Boyd, The sequence of radii of the Apollonian packing, Math. Comp. **39** (1982), 249–254.

[8] H. S. M. Coxeter, The problem of Apollonius, Amer. Math. Monthly **75** (1968), 5–15.

[9] H. S. M. Coxeter, *Introduction to Geometry, Second Edition*, John Wiley and Sons, New York, 1969.

[10] T. W. Cusick, Continuants with bounded digits, Mathematika **24** (1977), 166–172.

[11] I. Daubechies and J. C. Lagarias, Sets of matrices all infinite products of which converge, Lin. Alg. Appl. **161** (1992), 227–263; Corrigendum and Addendum, Lin. Alg. Appl. **327** (2001), 69–83.

[12] H. Davenport, *Multiplicative Number Theory. Third Edition. Revised and with and introduction by H. L. Montgomery,* Springer-Verlag: New York 2000.

[13] W. Duke, Notes on the distribution of points on $x^2+y^2+z^2 = w^2$, unpublished manuscript, Feb. 1993.

[14] K. J. Falconer, *The Geometry of Fractal Sets*, Cambridge Tracts in Math., vol. 85, Camb. Univ. Press, Cambridge, 1986.

[15] L. R. Ford, Fractions, Amer. Math. Monthly **45** (1938), 586–601.

[16] C. F. Gauss, *Disquisitiones Arithmeticae,* Leipzig 1801. (Reprinted in: Werke.) English translation: Springer-Verlag.

[17] R. L. Graham, J. C. Lagarias, C. L. Mallows, A. Wilks and C. Yan, Apollonian Packings: Geometry and Group Theory, I. Apollonian Group, `eprint: arXiv math.MG/0010298`

[18] R. L. Graham, J. C. Lagarias, C. L. Mallows, A. Wilks and C. Yan, Apollonian Packings: Geometry and Group Theory, II. Super-Apollonian Group and Integral Packings, `eprint: arXiv math.MG/0010302`

[19] R. L. Graham, J. C. Lagarias, C. L. Mallows, A. Wilks and C. Yan, Apollonian Packings: Geometry and Group Theory, III. Higher Dimensions, `eprint: arXiv math.MG/0010324`

[20] G. H. Hardy and E. M. Wright, *Introduction to the Theory of Numbers*, (4th ed.) Oxford University Press, 1960.

[21] K.E. Hirst, The Apollonian packing of circles, J. Lond. Math. Soc., **42** (1967), 281–291.

[22] A. Hurwitz, Solution to Problem 3084, *13* (1906), 164. In: *Mathematische Werke*, Vol II, Birkhäuser: Basle 1934, p. 751.

[23] E. Kasner and F. Supnick, The Apollonian packing of circles, Proc. Nat. Acad. Sci. USA **29** (1943), 378–384.

[24] J. C. Lagarias, C. L. Mallows and A. Wilks, Beyond the Descartes circle theorem, Amer. Math. Monthly **109** (2002), 338–361. `eprint: arXiv math.MG/0101066`

[25] E. Landau, *Vorlesungen über Zahlentheorie*, Vol. I, Teubner: Leipzig 1927. Translation: *Elementary Number Theory*, Chelsea: New York

[26] S. Lang, *Algebraic Number Theory*, (2nd ed.),1967 [Ch. VIII §2 Theorem 5, p. 161].

[27] D. G. Larman, On the exponent of convergence of a packing of spheres, Mathematika **13** (1966), 57–59.

[28] P. D. Lax and R. S. Phillips, The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces, J. Funct. Anal. **46** (1982), 280–350.

[29] B. B. Mandelbrot, *The Fractal Geometry of Nature*, Freeman: New York, 1982.

[30] G. B. Mathews, *Theory of Numbers*, Second Edition, Chelsea: New York 1961. (First edition: Cambridge 1892.)

[31] J. G. Mauldon, Sets of equally inclined spheres, Canad. J. Math. **14** (1962), 509–516.

[32] G. Maxwell, Sphere packings and hyperbolic reflection groups. J. Algebra **79** (1982), 78–97.

[33] Z. A. Melzak, Infinite packings of disks, Canad. J. Math. **18** (1966), 838–853.

[34] Z. A. Melzak, On the solid-packing constant for circles, Math. Comp. **23** (1969), 169–172.

[35] L. J. Mordell, *Diophantine Equations,* Academic Press: New York 1969.

[36] P. J. Nicholls, Diophantine approximation via the modular group, J. London Math. Soc. **17** (1978), 11–17.

[37] S. Northshield, On Apollonian circle packings, preprint.

[38] H. Rademacher, *Lectures on Elementary Number Theory,* Blaisdell: New York 1964.

[39] H. Rademacher, *Topics in Analytic Number Theory*, Springer-Verlag: Berlin 1973.

[40] J. G. Ratcliffe and S. T. Tschantz, On the representation of integers by the Lorentzian quadratic form, J. Funct. Anal. **150** (1997), 498–525.

[41] B. Rodin and D. Sullivan, The convergence of circle packings to the Riemann mapping, J. Differential Geometry **26** (1987), 349–360.

[42] T. Rothman, Japanese temple geometry, Scientific American, May 1998, 84–91.

[43] H. F. Sandham, A square as the sum of seven squares, Quart. J. Math. (Oxford) **4** (1953), 230–236.

[44] F. Soddy, The Kiss Precise, Nature **137** (June 20, 1936), 1021.

[45] F. Soddy, The bowl of integers and the Hexlet, Nature **139** (1937), 77–79.

[46] B. Söderberg, Apollonian tiling, the Lorentz group, and regular trees, Phys. Rev. A **46** (1992), No. 4, 1859–1866.

[47] E. C. Titchmarsh, *The Theory of the Riemann Zeta Function,* (Revised by D. R. Heath-Brown) Oxford, 1986.

[48] P. B. Thomas and D. Dhar, The Hausdorff dimension of the Apollonian packing of circles, J. Phys. A: Math. Gen. **27** (1994), 2257–2268.

[49] C. Tricot, A new proof for the residual set dimension of the Apollonian packing, Math. Proc. Cambridge Phil. Soc. **96** (1984), 413–423.

[50] A. I. Weiss, On isoclinal sequences of spheres, Proc. Amer. Math. Soc. **88** (1983), 665–671.

[51] J. B. Wilker, Open disk packings of a disk, Canad. Math. Bull., **10** (1967), 395–415.

[52] J. B. Wilker, Inversive Geometry, in: *The Geometric Vein*, (C. Davis, B. Grünbaum, F. A. Sherk, Eds.), Springer-Verlag: New York 1981, pp. 379–442.

[53] S. K. Zaremba, La methode des "bonnes treillis" pour le calcul des integrales multiples, in *Applications of number theory to numerical analysis,* (Montreal, 1971), (S. K. Zaremba, Ed.) Academic Press, New York, 1972, pp. 39–119.

email: graham@ucsd.edu
jcl@research.att.com
colinm@research.avayalabs.com
allan@research.att.com
Catherine.Yan@math.tamu.edu