Lens Sequences

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Abstract

A family of sequences produced by a non-homogeneous linear recurrence formula derived from the geometry of circles inscribed in lenses is introduced and studied. Mysterious "underground" sequences underlying them are discovered in this paper.

1 Introduction

We investigate a new family of integer sequences. They are generated by a geometric construction, which we now describe.

Start with three circles of curvatures a, b, and c centered on the same line, so that pairs of consecutive circles are tangent, as in the left side of Figure 1.1. The three circles determine a pair of congruent circles that are simultaneously tangent to the original triple (right side of Figure 1.1). The common region formed by this pair defines a symmetric **lens**. Now, continue to inscribe circles inside the lens, as shown in Fig. 1.1. The resulting chain of circles defines a bilateral sequence of curvatures (b_i), $i \in \mathbb{Z}$. Sequences obtained this way will be called **lens sequences**.

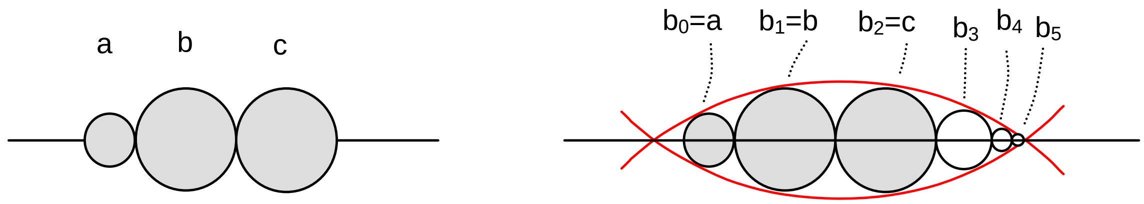

Figure 1.1: Three circles determine a sequence

Terminology. The resulting circles form a **lens circle chain**. We shall say that a triplet of circles (a, b, c) **generates** the lens sequence, and we will call it a **seed** of the sequence. Notice that *any* three consecutive terms of a lens sequence define a seed. The two circles that form the lens will be called **lens circles**.

Notation. Typically, we denote circles and their curvatures by the same letter (circle a has curvature a, i.e., radius $1/a$).

Our opening result (proved in the next section) is this:

Theorem A. *Let a, b and c be the curvatures of the initial three circles generating a lens sequence, $b \neq 0$. Then the sequence is determined by the following inhomogeneous three-term recurrence formula:*

$$b_n = \alpha\, b_{n-1} - b_{n-2} + \beta\,, \tag{1.1}$$

where α and β are constants determined by the original triple:

$$\alpha = \frac{ab + bc + ca}{b^2} - 1 \quad \textit{and} \quad \beta = \frac{b^2 - ac}{b}\,. \tag{1.2}$$

In particular, if $b_0 = a$ and $b_1 = b$ then $b_2 = c$.

Constants α and β are "invariants" of the sequence – their values may be determined from any three consecutive terms of the sequence.

If a, b, c as well as α and β are integers, then (b_n) is an integer sequence. Surprisingly, this family of sequences includes a wide range of known sequences [9]. However, for some of these sequences, properties that we develop in this paper seem to be new. For now, let us look at a few examples.

Example 1.1 (Vesica Piscis)**.** Starting with $(a, b, c) = (3,1,3)$ we get the recurrence formula

$$b_n = 14b_{n-1} - b_{n-2} - 8\,,$$

which produces

$$\ldots \mathbf{3,\ 1,\ 3},\ 33,\ 451,\ 6273,\ 87363,\ 1216801,\ 16947843,\ 236052993,\ \ldots,$$

a sequence known as A011922. Note that if one starts with values twice as large, the whole sequence is doubled:

$$\ldots \mathbf{6},\ \mathbf{2},\ \mathbf{6},\ 66,\ 902,\ 12546,\ 174726,\ 2433602,\ 33895686,\ 472105986,\ \ldots$$

and the recurrence formula becomes $b_n = 14b_{n-1} - b_{n-2} - 16$ (the same α and twice β). (The lens circles traverse each others center, forming a well-known figure of *Vesica Piscis*, hence the name of the example.)

Example 1.2 (Golden Vesica)**.** Start with $(a, b, c) = (1, 2, 10)$. Equations (4.2) give $\alpha = 7$ and $\beta = -3$; hence the sequence is generated by

$$b_n = 7b_{n-1} - b_{n-2} - 3$$

and is

$$\mathbf{1},\ \mathbf{2},\ \mathbf{10},\ 65,\ 442,\ 3026,\ 20737,\ 142130,\ 974170,\ \ldots$$

for positive n. This sequence, listed as Sloan's A064170, is known for its interesting properties. Its terms are products of pairs of non-consecutive Fibonacci numbers: $1 \cdot 2$, $2 \cdot 5$, $5 \cdot 13$, $13 \cdot 34$, $\ldots$, etc. They also coincide with the denominators in a system of Egyptian fraction for ratios of consecutive Fibonacci numbers: $1/2 = 1/\mathbf{2}$, $3/5 = 1/\mathbf{2} + 1/\mathbf{10}$, $8/13 = 1/\mathbf{2} + 1/\mathbf{10} + 1/\mathbf{65}$, etc. (The geometry of the lens relates to the golden proportion, hence the proposed name.)

Example 1.3. Triplet $(-1, 3, 15)$ gives

$$\ldots 99,\ 63,\ 35,\ 15,\ 3,\ \mathbf{-1},\ \mathbf{3},\ \mathbf{15},\ 35,\ 63,\ 99,\ 143,\ \ldots$$

from the recurrence

$$b_n = 2b_{n-1} - b_{n-2} + 8\,.$$

The sequence $(3,\ 15,\ 35,\ \ldots)$ is known as A000466 and is defined by $b_n = 4n^2 - 1$. The occurrence of negative curvatures will be explained later.

Example 1.4. A lens sequence does not necessarily need to be symmetric. For instance the triple (2,1,3) produces the following bilateral sequence:

$$\ldots,\ 12972,\ 1311,\ 133,\ 14,\ \mathbf{2},\ \mathbf{1},\ \mathbf{3},\ 24,\ 232,\ 2291,\ 22673,\ 224434,\ \ldots$$

Example 1.5 (More Examples)**.** The lens sequences possess an ample diversity. They include such basic examples as (i) the powers of 2 (A000079), and (ii) triangular numbers (A000217).

(i) **1, 2, 4**, 8, 16, 32, 64, 128, 256, ... $\qquad b_n = 5/2\, b_{n-1} - b_{n-2}$

(ii) **1, 3, 6**, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, ... $\qquad b_n = 2b_{n-1} - b_{n-2} + 1$

A more extensive list with references to OEIS [9] is provided in Tables 4.1 through 4.3.

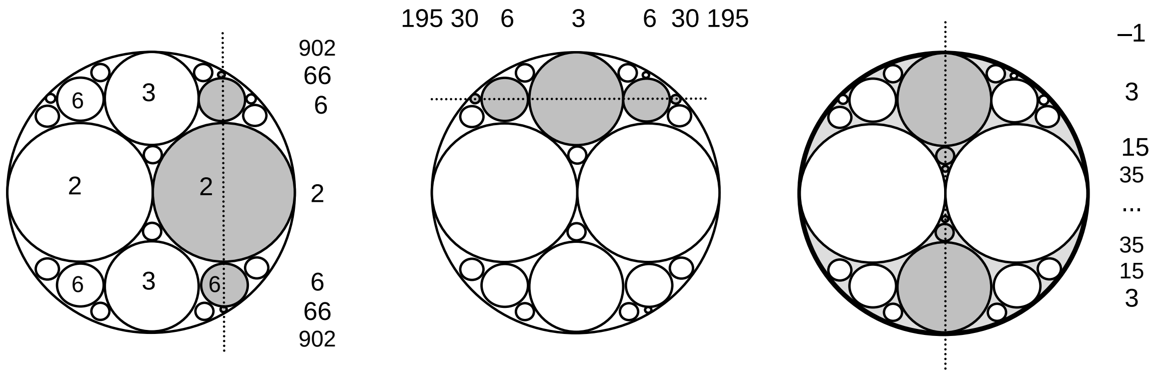

Figure 1.2: Sequences A011922, A064170, and A000466 in the Apollonian window

Remark 1.6. Certain circle packings, known as Apollonian gaskets [5], result in *integral* curvatures for all of the circles (see e.g., [4]). One such gasket, an *Apollonian Window* (see [2, 3]), is presented in Fig. 1.2. Interestingly, it contains an infinite number of lens sequences, from which the three shown in Fig. 1.2 are especially conspicuous. They correspond to the Examples 1.1, 1.2, and 1.3, given above (up to scaling). This observation was the author's initial motivation for this study.

The negative term of Example 1.3 (the third one in Figure 1.2) has a clear geometric meaning: it is the curvature of the greatest disc in the sequence, which — unlike the other discs in the sequence — contains the lens circles (as well as the rest of the sequence).

A number of interesting features are common to all lens sequences:

1. Limits. In many cases, the limit of the ratios of consecutive entries is well defined. For instance, referring to the above examples:

Example 1 [A011922]: $\lim\limits_{n\to\infty} \dfrac{b_{n+1}}{b_n} = 7 + 4\sqrt{3} = (2+\sqrt{3})^2$

Example 2 [A064170]: $\lim\limits_{n\to\infty} \dfrac{b_{n+1}}{b_n} = \dfrac{7+3\sqrt{5}}{2} = \left(\dfrac{1+\sqrt{5}}{2}\right)^4$

Example 3 [A000466]: $\lim\limits_{n\to\infty} \dfrac{b_{n+1}}{b_n} = 1$

These numbers are examples of Pisot numbers and will be called characteristic constants of the sequences denoted $\lambda = (\alpha + \sqrt{\alpha^2 - 4})$.

2. Sums. The reciprocals of curvatures are the circles' radii. Their sum is determined by the length of the lens. For instance:

Example 1 [A011922]: $\Sigma_{i=1}^{\infty} 1/b_i = 1 + 1/3 + 1/33 + 1/451 + \ldots = \sqrt{3}/2$

Example 2 [A064170]: $\Sigma_{i=1}^{\infty} 1/b_i = 1 + 1/2 + 1/10 + 1/65 + \ldots = (1+\sqrt{5})/2 = 1.618\ldots$

Example 3 [A000466]: $\Sigma_{n=1}^{\infty} 1/b_n = \sum_n 1/(n^2-1) = 1/3 + 1/15 + 1/35 + \ldots = 1$

3. Binet-type formulas. For $\alpha \neq 2$, the curvatures may be expressed in terms a non-homogeneous Binet-type formulas:

Example 1 [Vesica Piscis, A011922]: **3 1 3** 33 451 6273 87363 ...,

$$b_n = \frac{4 + (2+\sqrt{3})^{2n} + (2-\sqrt{3})^{2n}}{6}\,.$$

Example 2. [Golden Vesica [A064170]: ... **2 1 2** 10 35 442 ...,

$$b_n = \frac{3 + \left(\frac{1+\sqrt{5}}{2}\right)^{4n} + \left(\frac{1-\sqrt{5}}{2}\right)^{4n}}{5}$$

Example 3. Also non-symmetric lens sequences can be expressed this way. For instance, the sequence extended from $(6,2,3)$, which is (...2346 299 39 **6 2 3** 15 110 858 6747...), with the recurrence $b_n = 8b_{n-1} - b_{n-2} - 7$, may be obtained from

$$b_n = \frac{(25-3\sqrt{15})(4+\sqrt{15})^n + (25+3\sqrt{15})(4-\sqrt{15})^n}{60} + \frac{7}{6}$$

The above properties are known for some of the sequences, but now they acquire a geometric interpretation. Other related concepts include geometry of inversions, Chebyshev polynomials, etc.

The most remarkable and perhaps surprising property is that the integer lens sequences are "shifted squares" of yet deeper integer "underground" sequences. This discovery is the topic of the final section of this paper.

2 Recurrence formula from geometry

In this section we prove Theorem A on the recurrence formula for lens sequences. A reader interested in the algebraic properties of these sequences may skip it without loss of continuity.

We shall need a theorem on circle configurations generalizing that of Descartes' theorem on "kissing circles" ([1, 2]). If C_1 and C_2 denote two circles of radii r_1 and r_2 respectively, and d denotes the distance between their centers, then one defines a product of the circles as

$$\langle C_1, C_2 \rangle = \frac{d^2 - r_1^2 - r_2^2}{2r_1 r_2}\,, \tag{2.1}$$

which we propose to call the *Pedoe product*. Its values for a few cases are shown in Figure 2.1. For any four circles C_i, $i = 1, \dots, 4$, define a **configuration matrix** f as the matrix with entries

$$f_{ij} = \langle C_i, C_j \rangle ,$$

where the brackets denote the Pedoe inner product of circles.

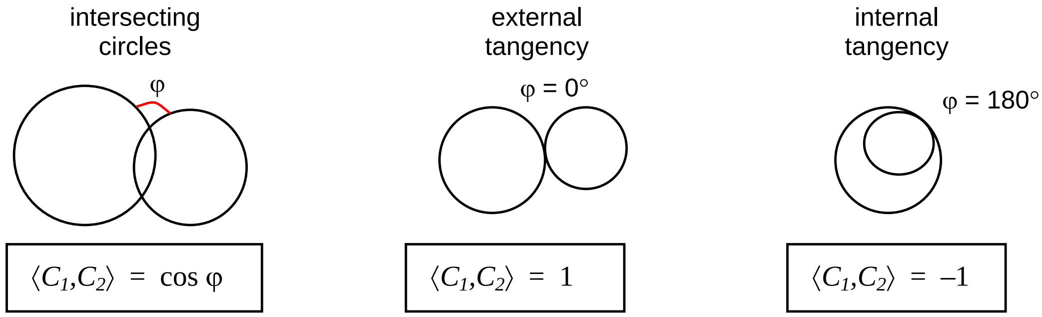

Figure 2.1: Pedoe product of circles

Theorem 2.1 ([2])**.** *A configuration of four circles in general position satisfies the following quadratic equation*

$$\boldsymbol{b}^T F \boldsymbol{b} = 0 \tag{2.2}$$

where $\boldsymbol{b} = [b_1, b_2, b_3, b_4]^T$ *is the vector made of the curvatures of the four circles, and where* $F = f^{-1}$ *is the inverse of the configuration matrix.*

Remark 2.2. Equation (2.2) is only a fragment of the full matrix formula, which incorporates also the positions of the centers of the circles. For more on this theorem, its proof, and the associated Minkowski geometry of circles, see [2]. For our purposes Formula (2.2) is sufficient.

Notation. In the following by a **lens** we mean "symmetric lens" — the intersection of the interiors (exteriors) of two congruent circles, called in this context **lens circles**. A **chain of circles** is a sequence of circles such that every two consecutive circles are tangent.

We are now ready to prove the basic result.

Theorem 2.3. *A sequence* (b_n) *of curvatures of a chain of circles inscribed in a lens satisfies a non-homogeneous linear recurrence formula of the form*

$$b_{n+1} = \alpha b_n - b_{n-1} + \beta$$

for some constants α *and* β*, with*

$$\alpha = \frac{6 - 2K}{1 + K} = \frac{8}{1 + K} - 2 \quad \text{and} \quad \beta = \frac{8A}{1 + K} , \tag{2.3}$$

where K *is the Pedoe product of the two lens circles and* $A = 1/R$ *is the curvature of each lens circle.*

Proof. Consider two consecutive circles in the lens, of curvatures say a and b. Denote the curvatures of the circles that form the lens by A, and their Pedoe product by K ($K = \cos\varphi$, if the circles intersect). The configuration matrix f and its inverse are easy to find. In the case of converging lenses we can read it off from Fig. 2.2a:

$$f = \left[\begin{array}{cc|cc} -1 & K & -1 & -1 \\ K & -1 & -1 & -1 \\ \hline -1 & -1 & -1 & +1 \\ -1 & -1 & +1 & -1 \end{array}\right]$$

where the indices are ordered as (A, A, x, y). Its inverse F is easy to find, and the master equation (2.1), after multiplying by a factor of 8, becomes:

$$\begin{bmatrix} A \\ A \\ x \\ y \end{bmatrix}^T \begin{bmatrix} \frac{4}{K+1} & \frac{-4}{K+1} & 2 & 2 \\ \frac{-4}{K+1} & \frac{4}{K+1} & 2 & 2 \\ 2 & 2 & K+1 & K-3 \\ 2 & 2 & K-3 & K+1 \end{bmatrix} \begin{bmatrix} A \\ A \\ x \\ y \end{bmatrix} = \quad 0$$

This quadratic equation is equivalent to:

$$(1+K)x^2 + (1+K)y^2 + 2(K-3)xy + 8Ax + 8Ay = 0\,.$$

One may solve it for y to get two solutions (corresponding to two signs at the square root):

$$y_{1,2} = \frac{4A + (K-3)x \pm 2\sqrt{2(1-K)x^2 - 8Ax + 4A^2}}{1+K} \tag{2.4}$$

Note that the two solutions y_1 and y_2 correspond to the two possible circles tangent to x: one on the left and one on the right. To eliminate radicals, add the two solutions:

$$y_1 + y_2 = \frac{6-2K}{1+K}x - \frac{8A}{1+K}\,.$$

Since the triple (y_1, x, y_2) forms a sequence in a chain of inscribed circles, we may label these curvatures as $b_{n-1} = y_1$, $b_n = x$, and $b_{n+1} = y_2$, to get

$$b_{n+1} + b_{n-1} = \alpha\, b_n + \beta\,,$$

which is equivalent to (2.3). The case of the diverging lens results from similar reasoning, with slightly different initial matrix F. □

Corollary 2.4. *The sequence constants are related:* $\alpha + R\,\beta = -2$.

Now let us see how three circles determine a sequence.

Theorem 2.5. *Let a, b and c be curvatures of three consecutive circles inscribed in a lens. Then the sequence of the circle curvatures is determined by the following three-term recurrence formula:*

$$b_n = \alpha\, b_{n-1} - b_{n-2} + \beta\,, \tag{2.5}$$

where $\alpha = \frac{ab+bc+ca}{b^2} - 1$ *and* $\beta = \frac{b^2-ac}{b}$*. If* $b_0 = a$ *and* $b_1 = b$ *then* $b_2 = c$*, that is,* $\alpha b + \beta = a + c$.

Proof. We apply Theorem 2.1 in each of the three steps to a different quadruple of circles.

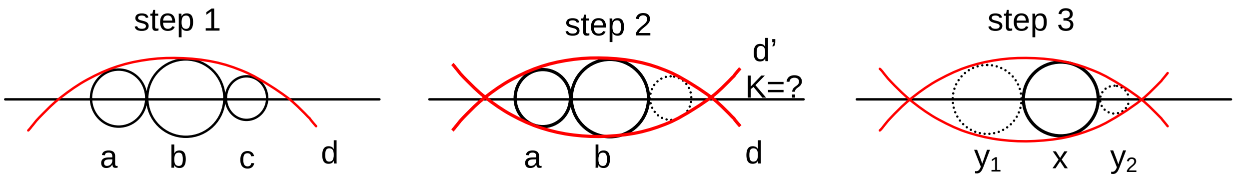

Figure 2.2: The three steps of the proof of Thm. 2.5

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Step 1. Consider a configuration of four circles: a triple of three consecutive circles in the chain, say a, b, c, plus one circle forming the lens, say $d = 1/R$. The Pedoe product of two external circles a and c may be easily evaluated; since the distance between their centers is $(1/a + 2/b + 1/c)$, we have:

$$\langle a, c\rangle = \frac{\left(\frac{1}{a}+\frac{2}{b}+\frac{1}{c}\right)^2 - \left(\frac{1}{a}\right)^2 - \left(\frac{1}{c}\right)^2}{2\cdot\frac{1}{a}\cdot\frac{1}{c}} = 2\frac{ab+bc+ca}{b^2}+1\,.$$

Denote the main fraction of the last expression by $z = \frac{ab+bc+ca}{b^2}$. Then the configuration matrix and its inverse are

$$f = \left[\begin{array}{ccc|c} -1 & 1 & 2z+1 & -1 \\ 1 & -1 & 1 & -1 \\ 2z+1 & 1 & -1 & -1 \\ \hline -1 & -1 & -1 & -1 \end{array}\right], \qquad F = \frac{1}{4}\begin{bmatrix} -\frac{1}{z+1} & 1 & \frac{1}{z+1} & -1 \\ 1 & -(z+1) & 1 & z-1 \\ \frac{1}{z+1} & 1 & -\frac{1}{z+1} & -1 \\ -1 & z-1 & -1 & -(z+1) \end{bmatrix}$$

(the order of entries is: $abcd$). Denoting $\mathbf{v} = [a, b, c, d]^T$ and solving the quadratic equation $\mathbf{v}^T F\mathbf{v} = 0$ for d readily leads to

$$d = \frac{b(ac-b^2)}{ab+bc+ca+b^2}\,.$$

This gives us the curvature of each of the two lens circles. Now we need to find the product of these two lens circles.

Step 2. Use a quadruple of circles: a, b, and the two circles forming the lens, d and d'. The latter two have the same curvature $d = d'$, the value of which we know from the previous step. The goal is to find the Pedoe product $K = \langle d, d'\rangle$. The configuration matrix and its inverse are

$$f = \begin{bmatrix} -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & K \\ -1 & -1 & K & -1 \end{bmatrix} \qquad F = \frac{1}{4}\begin{bmatrix} -1-K & 3-K & -2 & -2 \\ 3-K & -1-K & -2 & -2 \\ -2 & -2 & -\frac{4}{K+1} & \frac{4}{K+1} \\ -2 & -2 & \frac{4}{K+1} & -\frac{4}{K+1} \end{bmatrix}$$

(the order of indices agrees with $abdd'$). Applying vector $\mathbf{v} = [a, b, d, d]^T$ to the quadratic equation $\mathbf{v}^T F \mathbf{v} = 0$ gives

$$K = \frac{8b^2}{(a+b)(b+c)} - 1 \tag{2.6}$$

Step 3. Now we can either build the matrix for configuration (c) in Figure 2.2 and mimic the proof of Theorem 2.3, or simply substitute for K from (2.6) in (2.4) to get the result. □

3 More on lens geometry

Although we are mainly interested in the algebraic properties of lens sequences, some geometric properties explicate their algebraic behavior. Below, we summarize basic facts.

Proposition 3.1. The **radius** R of the lens circles is determined by three circles and may be expressed in terms of the sequence constants α and β:

$$R = \frac{\alpha+2}{-\beta} = \frac{(a+b)(b+c)}{(ac-b^2)b}\,. \tag{3.1a}$$

The Pedoe **inner product** of the lens circles is

$$K = \frac{6-\alpha}{2+\alpha} = \frac{8b^2}{(a+b)(b+c)} - 1 = \frac{1}{2}\left(\frac{\delta}{R}\right)^2 - 1 = \cos\varphi\,, \tag{3.1b}$$

where the last equation is valid if the circles intersect. The **length** L of the lens, if defined, is

$$L = 2R\sqrt{\frac{\alpha-2}{\alpha+2}} = -\frac{2\sqrt{\alpha^2-4}}{\beta} = 2\frac{\sqrt{(a+b)(b+c)[(a+b)(b+c)-4b^2]}}{(ac-b^2)b}\,. \tag{3.1c}$$

The **separation** of the lens circles (distance between their centers) is

$$\delta = \frac{4R}{\sqrt{\alpha+2}} = -\frac{4\sqrt{\alpha+2}}{\beta} \quad \text{and} \quad \frac{\delta}{R} = \frac{4}{\sqrt{\alpha+2}}\,. \tag{3.1d}$$

Proof. All are direct corollaries of Theorem 2.3 and simple geometric constructions. □

Figure 3.1 contains these findings for easy reference.

Figure 3.2 categorizes a variety of geometric situations for a lens sequence. In the case of converging lenses, when two circles of radius R intersect at angle φ, the recurrence formula is:

$$b_n = \left(\frac{8}{1+\cos\varphi} - 2\right) b_{n-1} - b_{n-2} - \frac{1}{R}\frac{8}{1+\cos\varphi}\,.$$

This answers the question of which lenses may lead to integer sequences. Indeed, denote $n = \frac{8}{1+\cos\varphi}$. Then $\alpha = n-2$, $\beta = -n/R$. For n to be an integer, $n \in \mathbb{N}$, we need $\cos\varphi = 8/n - 1$. Table 3.1 shows some values.

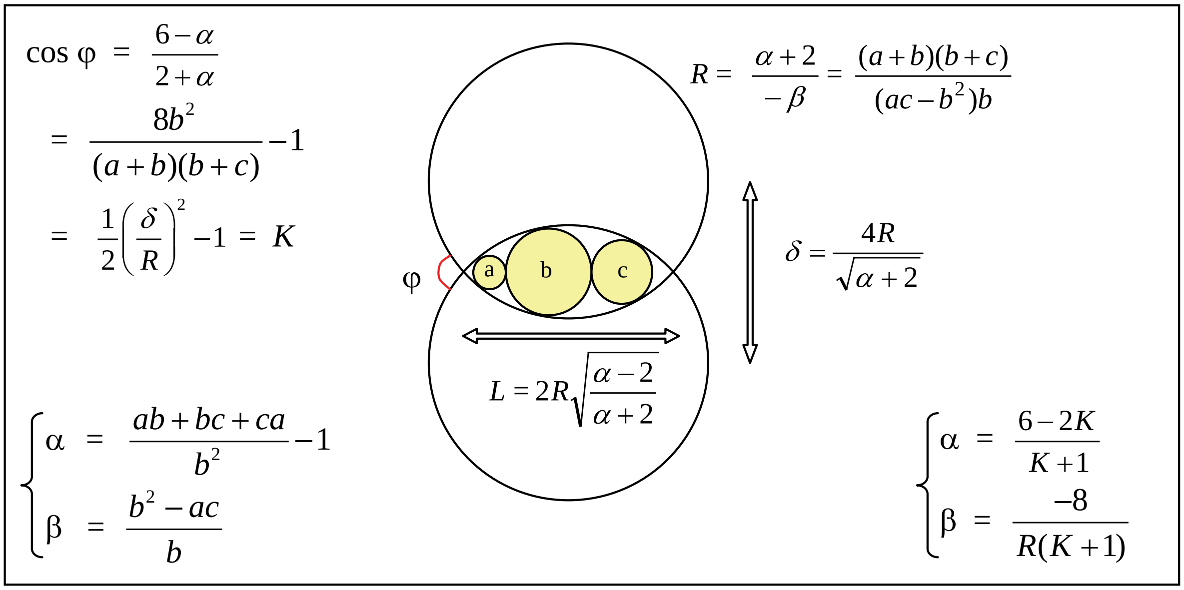


Figure 3.1: Sequence constants and geometry of a lens

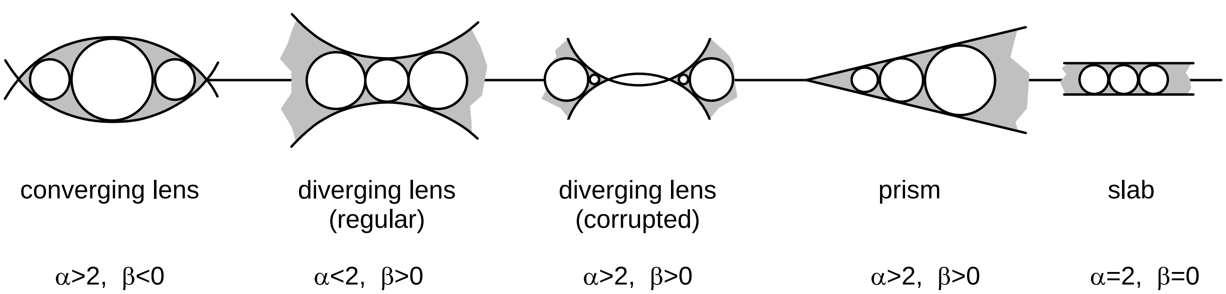


Figure 3.2: Types of lenses and associated sequences

Figure 3.2 relates the geometry of lenses to the values of the sequence constant α and the Pedoe product K. Inspect Tables 4.1 through 4.3 (and associated figures) in the next section for various examples of lens sequences.

Note that if $\alpha \leqslant 2$, then only external sequences are possible (corresponding to diverging lenses). Moreover, if $\alpha < 2$, then the integer sequence must be periodic. If $\alpha > 2$, then we can have two families of sequences: inner (inside a converging lens) or outer (outside the lens circles, i.e., inside a corrupted diverging lens (*corrupted*, because of the missing central part)). In the case of the outer sequence we will have exactly one negative entry (the most external circle) or two adjacent "0" entries (two vertical lines).

n	K	α		example	
0	∞	-2	periodic (order 2)	$(1,-1)$	
1	7	-1	periodic (order 3)	$(6,3,-2)$	disjoint lens circles
2	3	0	periodic (order 4)	$(3,6,2,-1)$	
3	$5/3$	1	periodic (order 6)	$(2,10,15,12,4,-1)$	
4	1	2	Example 1.3	$(-1,3,15)$	$\longleftarrow$ tangent lens circles
5	$3/4$	3		$(3,2,2,3)$	
6	$1/3$	4		$(2,1,1,2)$	
7	$1/7$	5		$(5,2,2,5)$	
8	0	6	orthogonal	$(3,1,1,3)$	
9	$-1/9$	7	golden Vesica	$(2,1,2)$	
10	$-1/5$	8		$(4,1,1,4)$	
11	$-3/11$	9			
12	$-1/3$	10		$(3,1,2)$	intersecting lens circles
13	$-5/3$	11			
14	$-3/7$	12		$(6,1,1,6)$	
15	$-7/15$	13		$(4,1,2)$	
16	$-1/2$	14	Vesica Piscis	$(3,1,3)$	
17	$-9/17$	15			
18	$-5/9$	16		$(8,1,1,8)$	
20	$-3/5$	17			

Table 3.1: Admissible values of α

4 Basic algebraic properties of lens sequences

Theorem A suggests the following definition:

Definition 4.1. A **formal sequence** extended from a triplet (a,b,c), called a **seed**, is defined by the following inhomogeneous three-term recurrence formula:

$$b_n = \alpha\, b_{n-1} - b_{n-2} + \beta\,, \tag{4.1}$$

where α and β are constants determined by the original triple:

$$\alpha = \frac{ab+bc+ca}{b^2} - 1 \quad \text{and} \quad \beta = \frac{b^2-ac}{b}\,. \tag{4.2}$$

and $b_0 = a$ and $b_1 = b$. (It follows that $b_2 = c$).

The values of the constants α and β do not depend on the particular choice of the triplet of consecutive terms (seed). Moreover, if $\alpha > -2$, then the sequence may be interpreted in terms of a chain of circles inscribed in a lens made by two disks each of curvature $R^{-1} = -\beta/(\alpha+2)$ separated by distance $\delta = 4R/\sqrt{\alpha+2}$. Formal integer lens sequences exist also for $\alpha < -2$ (see Table 4.4 for examples), but in such a case the geometric interpretation is unclear as the distance between the lens circles becomes imaginary.

In general, lens sequences take real values. However, if any three terms of a lens sequence are rational, so is the whole sequence. The question whether a particular seed produces an integer sequence will be answered for now this way:

Proposition 4.2 (Integrality Criterion 1)**.** If $b|ac$ and $b^2|(ab+bc+ca)$, for any $a,b,c \in \mathbb{N}$, then a lens sequence extended from (a,b,c) consists of integers.

Here are basic properties of lens sequences:

Proposition 4.3. Let (b_n) be a lens sequence. Then the following holds:

(i) Sequence (b_n) satisfies a homogeneous 4-term linear recurrence formula

$$b_n = (\alpha+1)b_{n-1} - (\alpha+1)b_{n-2} + b_{n-3}\,. \tag{4.3}$$

(ii) If $\alpha \geq 2$ then the sum of the reciprocals converges and equals:

$$\sum_{n=-\infty}^{\infty} \frac{1}{b_n} = \frac{2\sqrt{\alpha^2-4}}{-\beta} = \frac{L}{2}\,. \tag{4.4}$$

(iii) If α >2 then the limit of the ratios of consecutive terms exists and equals:

$$\lambda = \frac{\alpha+\sqrt{\alpha^2-4}}{2} = \frac{\sqrt{\alpha+2}+\sqrt{\alpha-2}}{2}\,. \tag{4.5}$$

(iv) If $\alpha \neq 2$, then the lens sequence generated from a seed (a,b,c) has the following Binet-like formula

$$b_n = w\lambda^n + \bar{w}\bar{\lambda}^n + \gamma \tag{4.6}$$

where

$$\lambda = \frac{\alpha+\sqrt{\alpha^2-4}}{2} \qquad \bar{\lambda} = \frac{\alpha-\sqrt{\alpha^2-4}}{2}.$$

and where

$$w = \frac{a-2b+c}{2(\alpha-2)} + \frac{c-a}{2(\alpha^2-4)}\sqrt{\alpha^2-4}, \qquad \gamma = \frac{-\beta}{\alpha-2}$$

and $\bar{w}$ and $\bar{\lambda}$ denote conjugates of w and λ in $\mathbb{Q}(\sqrt{\alpha^2-4})$, respectively. In particular, $(a,b,c) = (b_{-1}, b_0, b_1)$.

Proof. (i) Elementary. (ii) From the geometry of lenses, cf. (3.1c). See also Figure 4.1. (iii) Divide the recurrence formula by b_{n-1} to get

$$b_n/b_{n-1} = \alpha - b_{n-2}/b_{n-1} + \beta/b_{n-1}\,.$$

For large values of n, since the sequence is divergent, the last term becomes irrelevant and the equation becomes $\lambda = \alpha - 1/\lambda$, or simply

$$\lambda^2 - \alpha\lambda + 1 = 0\,, \tag{4.7}$$

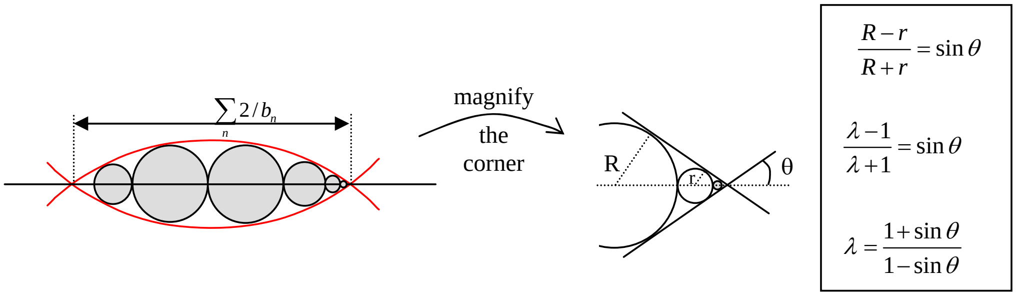


Figure 4.1: Geometric meaning of $\sum_n 1/b_n$ and λ (see Proposition 4.3)

with the solution as above. Figure 4.1 provides the geometric insight, which also relates λ to the lens angle via similar triangles. (iv) Define a new sequence whose entries are shifted by a constant, namely $a_n = b_n + \beta/(\alpha - 2)$. The sequence (a_n) satisfies a homogeneous three-term recurrence formula $a_n = \alpha\, a_{n-1} - a_{n-2}$, which resolves to 4.6 by the standard procedure. □

The value of λ (given by (4.5)) will be called the **characteristic constant** of the sequence. The ring over rational numbers generated by $\sqrt{\alpha^2-4}$ plays an important role in other properties of lens sequences, as we shall soon see. Note that the sequence constant α may be expressed in terms of the characteristic constant in a graceful way:

$$\alpha = \lambda + \frac{1}{\lambda}$$

Alternative generating formulae

Note that the three term formula (4.1), with given coefficients α and β, requires only two initial entries to produce a sequence. Yet not *all* such initial values will produce a lens sequence of the type under discussion. This is because arbitrary initial values b_0 and b_1 do not need to be geometrically inscribable into a lens defined by α and β as two consecutive circles. The following will clarify the situation:

Proposition 4.4 (Compatibility condition)**.** Two consecutive circles a and b in a lens chain satisfy the following condition:

$$a^2 + b^2 = \alpha ab + \beta(a+b)\,. \tag{4.8}$$

Proof. Eliminate c from the expressions for α and β in (4.2), and simplify. □

The above formula may actually be used as an alternative definition of lens sequences. Indeed,

Proposition 4.5. Consider the following properties:

(a) Recurrence $b_{n+1} = \alpha b_n - b_{n-1} + \beta$, for all n,

(b) Constants $\alpha = \frac{b_{n-1}b_n + b_n b_{n+1} + b_{n+1}b_{n-1}}{b_n 0^2} - 1$ and $\beta = \frac{b_n^2 - b_{n-1}b_{n+1}}{b_n}$.

(c) $a^2 + b^2 = \alpha\, ab + \beta(a+b)$, for $a = b_n$ and $b = b_{n+1}$.

The following descriptions of a sequence (b_i) are equivalent:

(i) Recurrence (a), and constants (b) for some n (definition of a lens sequence);

(ii) Recurrence (a) and compatibility condition (c) for some n;

(iii) Any of the two constant formulas (b) for all n;

(iv) Compatibility condition (c) for all n.

Proof. Let us show that the compatibility condition together with the recurrence theorem imply our standard formulas for both α and β. Starting with (ii) we get

$$\begin{aligned} b_{n+1}^2 + b_n^2 &= \alpha\, b_n b_{n+1} + \beta(b_n + b_{n+1}) \\ &= b_{n+1}(\alpha\, b_n + \beta) + \beta b_n \\ &= b_{n+1}(b_{n+1} + b_{n-1}) + \beta b_n\,. \end{aligned}$$

Subtracting b_{n+1}^2 from both sides, we get $b_n^2 = b_{n+1}b_{n-1} + \beta\, b_n$, which gives

$$\beta = \frac{b_n^2 - b_{n-1}b_{n+1}}{b_n}\,.$$

To get α, substitute this result in the compatibility condition and simplify. The other equivalences follow easily. □

Here is yet another intriguing formula that will prove itself handy later.

Proposition 4.6. Constant α has an alternative form involving any four consecutive entries of a lens sequence:

$$\alpha = \frac{b_{n-1}}{b_n} + \frac{b_{n+2}}{b_{n+1}}\,. \tag{4.9}$$

Proof. Start with the formula for β and express it as follows:

$$\beta = \frac{b_n^2 - b_{n+1}b_{n-1}}{b_n} = b_n - \frac{b_{n+1}b_{n-1}}{b_n} \quad \Rightarrow \quad b_n - \beta = \frac{b_{n+1}b_{n-1}}{b_n}\,.$$

Use the recurrence formula to modify the left-hand side of the last equation,

$$\alpha\, b_{n+1} - b_{n+2} = \frac{b_{n+1}b_{n-1}}{b_n}\,.$$

Now extract α to get 4.9. □

Integrality condition

Lens sequences are self-generating in the sense that any three consecutive entries (a, b, c), a seed, determine the whole sequence (unless $b = 0$). The question is how to choose seeds (a, b, c) in order to obtain lens sequences that are *integer*. Below we give only a partial answer to this problem of integrality conditions; the last section will provide the solution.

Definition 4.7. An integer lens sequence is *primitive* if the common divisor of three consecutive entries is 1.

Proposition 4.8. If $\gcd(b_k, b_{k+1}, b_{k+2}) = n$ holds for some k, then it holds for all $k \in \mathbb{Z}$.

Proof. If n divides each of (b_k, b_{k+1}, b_{k+2}), then it divides β of the recurrence formula. Hence it divides the neighboring terms b_{k+2} and b_{k-1}. By induction, n divides every term of the sequence. □

Recall that to insure that α and β are integers, we need to choose (a, b, c) so that $b|ac$ and $b^2|ab + bc + ca$, or, equivalently, that $b^2|(a + b)(b + c)$. Thus triples of the form $(a, 1, c)$ always generate integer sequences for any $a, c \in \mathbb{N}$. For now, let us review the following families of integer lens sequences:

A. A sequence is called **central** if it contains a triple of the form (a, b, a). Only $b = \pm 1$ leads to primitive integer sequences.

B. A lens sequence is called **bicentral**, if it contains a quadruplet of the form (a, b, b, a). Only if b is chosen from $\{0, 1, 2\}$, does a primitive integer sequence result. (For the case $b = 0$, the seed needs to be chosen in the form $(0, a, c)$).

In either case A or case B, the sequence is called **symmetric**. Table 4.1 shows examples of symmetric lenses for small values of the initial terms. (Only the right tail is displayed.)

C. Here is a method of getting a not-necessarily symmetric integer sequence: choose arbitrarily a couple (a, b) and some integer k (only one of a or b can be negative). Then a seed (a, b, c) with

$$c = b(bk - 1)\,, \tag{4.10}$$

will generate an integer sequence. Indeed, calculate the recurrence constants α and β from (4.10) to get the integer values

$$\begin{aligned} \alpha &= (a + b)k - 2 \\ \beta &= (a + b) - ab\,k\,. \end{aligned} \tag{4.11}$$

The triple of integers $[a, b; k]$ will be called the **label** of a lens sequence of this type. (Note that it includes the symmetric lens sequences as a special case; for examples of non-symmetric sequences, see Table 4.2.)

The pair (a, b) in the label may be chosen so that it contains the smallest element of the sequence.

	central elements	constants	OEIS nr	sequence	label and symbol	
Central sequences						
1.	$(2,1,2)$	$\alpha=7,\beta=-3$	A064170	$1,2,10,65,442,3026,20737,142130,\dots$	[1,2;3]	${}^3(1,1)^3$
2.	$(3,1,3)$	$\alpha=14,\beta=-8$	A011922	$1,3,33,451,6273,87363,1216801,16947843,\dots$	[1,3;4]	${}^4(1,1)^4$
3.	$(4,1,4)$	$\alpha=23,\beta=-15$	—	$1,4,76,1729,39676,910804,20908801,\dots$	[1,4;5]	${}^5(1,1)^5$
4.	$(5,\ 1,\ 5)$	$\alpha=34,\beta=-24$	—	$1,5,145,4901,166465,5654885,192099601,\dots$	[1,5;6]	${}^6(1,1)^6$
5.	$(3,-1,3)$	$\alpha=2,\beta=8$	A000466	$-1,3,15,35,63,99,143,195,255,323,399,\dots$	$[-1,3;2]$	${}^2(1,3)^2$
6.	$(4,-1,4)$	$\alpha=7,\beta=15$	A081078	$-1,4,44,319,2204,15124,103679,710644,\dots$	$[-1,4;3]$	${}^3(1,4)^3$
7.	$(5,-1,5)$	$\alpha=14,\beta=24$	—	$-1,5,95,1349,18815,262085,3650399$	$[-1,5;4]$	${}^4(1,5)^4$
8.	$(6,-1,6)$	$\alpha=23,\beta=35$	—	$-1,6,174,4031,92574,2125206,48787199,\dots$	$[-1,6;5]$	${}^5(1,6)^5$
Bicentral sequences						
9.	$(2,1,1,2)$	$\alpha=4,\beta=-1$	A101265	$1,2,6,21,77,286,1066,3977,14841,\dots$	[1,1;3]	${}^2(1,1)^3$
10.	$(3,1,1,3)$	$\alpha=6,\beta=-2$	A011900	$1,3,15,85,493,2871,16731,97513,568345,\dots$	[1,1;4]	${}^2(1,1)^4$
11.	$(4,1,1,4)$	$\alpha=8,\beta=-3$	—	$1,4,28,217,1705,13420,105652,831793,\dots$	[1,1;5]	${}^2(1,1)^5$
12.	$(5,1,1,5)$	$\alpha=10,\beta=-4$	A054318	$1,5,45,441,4361,43165,427285,4229681,\dots$	[1,1;6]	${}^2(1,1)^6$
13.	$(3,2,2,3)$	$\alpha=3,\beta=-1$	A032908	$2,3,6,14,35,90,234,611,1598,4182,10947,\dots$	[2,3;1]	${}^5(1,2)^1$
14.	$(5,2,2,5)$	α=5, β= -3	—	$2,5,20,92,437,2090,10010,47957,229772,\dots$	[2,5;1]	${}^7(1,2)^1$
15.	$(0,0,1,3)$	$\alpha=2,\beta=1$	A000217	$0,1,3,6,10,15,21,28,36,45,55,66,\dots$	[0,1;4]	${}^1(1,1)^4$
16.	$(0,0,1,4)$	$\alpha=3,\beta=1$	A027941	$0,1,4,12,33,88,232,609,1596,4180,\dots$	[0,1;5]	${}^1(1,1)^5$
17.	$(0,0,1,5)$	$\alpha=4,\beta=1$	A061278	$0,1,5,20,76,285,1065,3976,14840,55385,\dots$	[0,1;6]	${}^1(1,1)^6$

Table 4.1: Examples of symmetric lens sequences

	seed	constants	sequence	label and symbol	
1a.	$(3,1,2)$	$\alpha=10,\ \beta=-5$	24, 3, 1, 2, 14, 133, 1311, 12972, 128404, 1271063,...	[1,2;4]	${}^3(1,2)^4$
b.	$(2,1,3)$		14, 2, 1, 3, 24, 232, 2291, 22673, 224434, 2221662,...	[1,3;3]	${}^4(1,3)^3$
2.	$(5,3,6)$	$\alpha=6,\ \beta=-7$	$\dots$, 108, 20, 5, 3, 6, 26, 143, 825, 4800, 27968, ...	[5,3;1]	${}^8(1,3)^1$
3.	$(3,-1,4)$	$\alpha=4,\ \beta=11$	$\dots$, 403, 104, 24, 3, -1, 4, 28, 119, 459, 1728,$\dots$	$[-1,4;2]$	${}^3(1,4)^2$
4.	$(15,12,20)$	$\alpha=4,\ \beta=-13$	$\dots$, 400, 112, 35, 15, 12, 20, 55, 187, 680, 2520, ...	—	${}^2(4,5)^3$
5.	$(21,6,10)$	$\alpha=10,\ \beta=-29$	$\dots$, 16796, 1700, 175, 21, 6, 10, 65, 611, 6016, $\dots$	—	${}^4(2,5)^3$
6.	$(1,2,4)$	$\alpha=5/2,\ \beta=0$	$\dots$, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, $\dots$	[1,2;3/2]	${}^3(1,2)^{2/3}$

Table 4.2: Examples of non-symmetric sequences. Example 6 (A000079) is integer in one direction only.

	seed	constants	sequence	label and symbol	
1.	(2, -1,2)	$\alpha=-1,\ \beta=3$	2, 2, -1, 2, 2, -1, 2, 2, -1, 2, 2, -1, 2, 2, $\dots$	$[-1,2;1]$	${}^1(1,2)^1$
2.	(3, -1, 2)	$\alpha=0,\ \beta=5$	2, 6, 3, -1, 2, 6, 3, -1, 2, 6, -1, 2, 6,-1, $\dots$	$[-1,2;2]$	${}^1(1,2)^2$
3.	(14, -6, 15)	$\alpha=0,\ \beta=29$	14, -6, 15, 35, 14, -6, 15, 10, 35, 14, -6 $\dots$	—	${}^2(5,7)^1$
4.	(1, 1, 0)	$\alpha=0,\ \beta=1$	1, 1, 0, 0, 1, 1, 0, 0, 1, 1,0, 0, 1, 1, 0, 0, 1, 1, $\dots$	[0,1;2]	${}^1(1,1)^2$
5.	(4, -1, 2)	$\alpha=1,\ \beta=7$	2, 10, 15, 12, 4, -1, 2, 10, 15, 15, 12, 4, -1 $\dots$	$[-1,2;3]$	${}^1(1,2)^3$
6.	(10, -6, 33)	$\alpha=1,\ \beta=49$	33, 88, 104, 65, 10, -6, 33, 88, 104, 65, 10, -6, ...	—	${}^3(5,2)^1$

Table 4.3: Examples of periodic sequences. Example 4 is known as A021913.

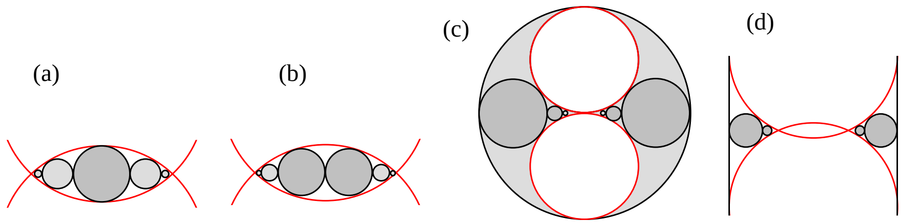


Figure 4.2: Geometric representations (only general shape) for examples from Table 4.1: (a) Examples 1–4, (b) Examples 9–14, (c) Examples 5–8, (d) Examples 15–17.

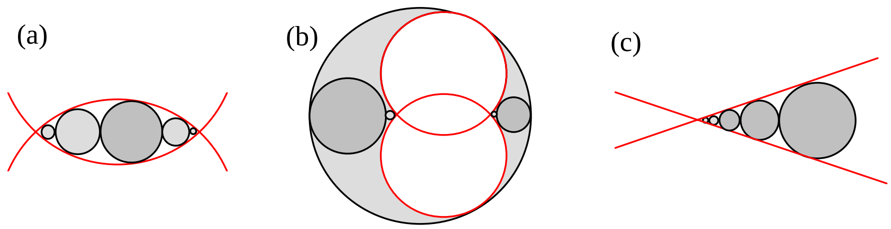


Figure 4.3: Geometric representations for examples from Table 4.2: (a) Examples 1–2 (b) Example 3, (c) Example 4.

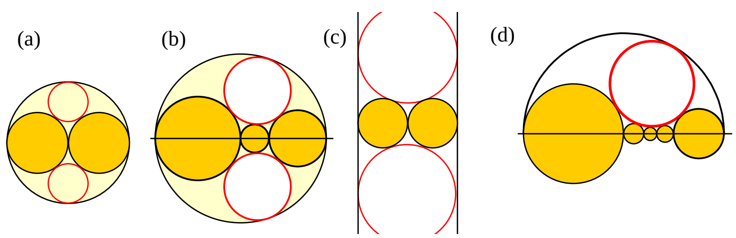


Figure 4.4: Geometric representations for examples from Table 4.3: (a) Example 1, (b) Example 2, (c) Example 3, (d) Example 4.

	constants	OEIS	sequence	label and symbol	
1.	$\alpha = -3,\ \beta = 1$	A001654	$0, 1, -2, 6, -15, 40, -104, 273, -714, 1870, -4895, 12816, \ldots$	[0,1;–1]	$^{1}(1,1)^{-1}$
2.	$\alpha = -3,\ \beta = 5$	A075269	$2, -3, 12, -28, 77, -198, 522, -1363, 3572, -9348, \ldots$	[2,2;1]	$^{-1}(1,2)^{1}$
3.	$\alpha = -4,\ \beta = 3$	—	$1, -2, 10, -35, 133, -494, 1846, -6887, 25705, -95930, \ldots$	[1,1;–1]	$^{2}(1,1)^{-1}$
4.	$\alpha = -4,\ \beta = -1$	A109437	$0, -1, 3, -12, 44, -165, 615, -2296, 8568, -31977, \ldots$	[0,–1;2]	$^{2}(1,3)^{-1}$
5.	$\alpha = -5,\ \beta = 1$	A099025	$0, 1, -4, 20, -95, 456, -2184, 10465, -50140, 240236, \ldots$	[0,1;–3]	$^{-3}(1,1)^{1}$
6.	$\alpha = -6,\ \beta = -4$	A084159	$1, -3, 21, -119, 697, -4059, 23661, -137903, 803761, \ldots$	[1,1;–2]	$^{-2}(1,1)^{-2}$
6.	$\alpha = -6,\ \beta = 1$	A084158	$0, 1, -5, 30, -174, 1015, -5915, 34476, -200940, \ldots$	[0,1;–4]	$^{-4}(1,1)^{1}$

Table 4.4: Examples of formal lens sequences

But the above types of sequences do not exhaust the possibilities, as this example shows:

$$\ldots\ 2331,\ 407,\ 77,\ 21,\ 15,\ 35,\ 161,\ 897,\ 5187\ \ldots\ ,$$

which is a lens sequence with recurrence formula $b_n = 6b_{n-1} - b_{n-2} - 34$. We will arrive at a general rule that produces all integer lens sequences and an improved version of the integrality criterion in the last section.

Invariants

The formulas for the coefficients α and β in the recurrence formula (4.1) may be represented diagrammatically as shown in Figure 4.5, which exhibits the coefficients' algebraic "structure" (the mnemonic role aside). The dots on the line represent the consecutive terms of the sequence. The arcs represent products of the joined terms, and the position above/below the line position indicates their appearance in the numerator/denominator of the formula. Dotted lines are to be taken with the negative sign.

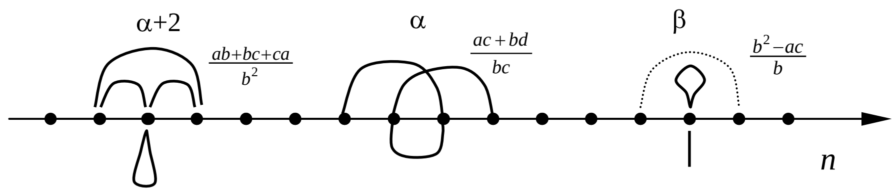


Figure 4.5: Diagrammatic representation of the lens sequence invariants

Since the formulas do not depend on the particular choice of the three seed circles, one may position the diagram at any place in the line/sequence. In this sense, α and β represent **invariants** of the sequence with respect to translation along the sequence. But this also means that each of them gives rise to a new non-linear recurrence formula!

Additional remarks on the Binet-like formula

Denoting "jumps" around the central element b_0 by $\Delta_+ = b_1 - b_0$ and by $\Delta_- = b_0 - b_{-1}$, we get a more suggestive form of term w in the Binet-like formula for lens sequences, namely:

$$w = \frac{\Delta_+ - \Delta_-}{2(\alpha-2)} + \frac{\Delta_+ + \Delta_-}{\alpha^2-4}\sqrt{\alpha^2-4}\,.$$

Expressing α in terms of λ we also obtain

$$w = \frac{\Delta_+ - \Delta_-}{2(\lambda+\lambda^{-1}-2)} + \frac{\Delta_+ + \Delta_-}{\sqrt{\lambda^2+\lambda^{-2}}}\,.$$

Other representations of the formula for w include:

$$w = \left(\frac{b_0}{2} + \frac{\beta}{\alpha-2}\right) + \frac{b_1 - b_{-1}}{2\alpha^2-8}\sqrt{\alpha^2-4}\,,$$

$$w = \frac{(\alpha+2)(b_1 - 2b_0 + b_{-1}) \quad + \quad (b_1 - b_{-1})\sqrt{\alpha^2-4}}{2(\alpha^2-4)}\,.$$

Note that re-indexing the sequence so that another term of (b_n) becomes the central "b_0" will change the value of w in the formula (4.6).

In order to better understand the situation, note the following simple general rule for this type of recurrences:

Theorem 4.9. *Let a sequence (x_n) be given by the following Binet-like formula*

$$x_n = a\omega^n + b\omega^{-n} + c \tag{4.12}$$

for some constants a, b, c and ω. Then the sequence satisfies a non-homogeneous 3-term linear recurrence formula

$$x_{n+2} = (\omega + 1/\omega)x_{n+1} - x_n + c(2 - \omega - 1/\omega)\,.$$

Moreover, if $|\omega| > 1$ then $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \omega$, and for large n we have $x_n \simeq a\omega^n + c$.

Proof. Direct. First note that (4.12) for the $(n+2)$-nd term gives

$$x_{n+2} = a\omega^2\omega^n + b\omega^{-2}\omega^{-n} + c\,.$$

Adding these two gives, after some simple algebraic operations:

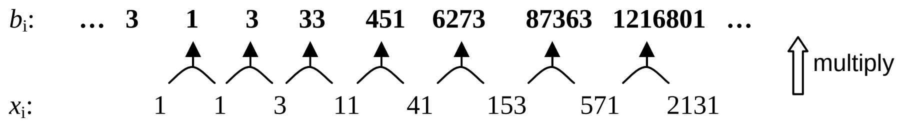
$$x_n + x_{n+2} = (\omega + 1/\omega)x_{n+1} + c(2 - \omega - 1/\omega)$$

which is equivalent to (4.12). ∎

Looking at the above, one hardly escapes the thought that lens sequences could be "explained" in terms of Chebyshev polynomials. This path did not however return any deeper insight. Instead, consider the following.

5 Underground sequences

There are still some mysteries in the structure of lens sequences. One of the most remarkable properties is this: the entries of an integer lens sequence are products of consecutive pairs of a certain "underlying" integer sequence. Here is an example of a sequence of *Vesica Piscis* (A011922, see Example 1.1 in Section 1):



Thus the lens sequence may be represented as $b_i = f_{i-1}f_i$, for some integer sequence (f_i). Sequence (f_i) will be called in this context the **underground** sequence of sequence (b_i) (Table 6.1 provides examples). We present the formalism of this amazing and unexpected property.

Let us start with a general fact.

Theorem 5.1. *Any factorization $\{f_n\}$ of lens sequence in a sense that $b_n = f_{n-1}f_n$ satisfies 3-term recurrence formula*

$$f_{n+2} + f_{n-2} = \alpha f_n \tag{5.1}$$

Proof. Starting with the expression for α given in Proposition 3.1, we have

$$\alpha = \frac{b_{n-1}}{b_n} + \frac{b_{n+2}}{b_{n+1}} = \frac{f_{n-2}f_{n-1}}{f_{n-1}f_n} + \frac{f_{n+1}f_{n+2}}{f_nf_{n+1}} = \frac{f_{n-2}}{f_n} + \frac{f_{n+2}}{f_n} = \frac{f_{n-2} + f_{n+2}}{f_n}$$

□

It should be borne in mind that this property is true for *any* —not necessarily integer— factorization of $\{b_i\}$. Such factorizations are easy to produce, e.g., set $f_0 = 1$, $f_1 = b_1$, $f_2 = b_2/b_1$, $f_3 = b_3b_1/b_0$, $f_4 = b_4b_2b_0/b_3b_1$, etc. However:

Theorem 5.2 (Factorization theorem)**.** *Any integer lens sequence (b_n) may be factored into an integer sequence (f_n) so that $b_n = f_{n-1}f_n$. If the lens sequence is primitive, the factorization is —up to a sign —unique. Moreover, in such a case $|f_n| = \gcd(b_n, b_{n+1})$.*

Proof. Assume that (b_n) is a primitive lens sequence. Consider three consecutive terms (a, b, c) and define

$$f_0 = \frac{a}{\gcd(a,b)} \quad f_1 = \gcd(a,b) \quad f_2 = \frac{b}{f_1} = \frac{b}{\gcd(a,b)} \quad f_3 = \frac{c}{f_2} = \frac{\gcd(a,b)c}{b}$$

Clearly, $a = f_0f_1$, $b = f_1f_2$, and $c = f_2f_3$. We need to show that these four terms are integers. Terms f_0, f_1, and f_2 are integer by definition. As to the last term, use the formula $\beta = \frac{b^2-ac}{b}$:

$$\beta \in \mathbb{Z} \quad \Rightarrow \quad \frac{ac}{b} \in \mathbb{Z} \quad \Rightarrow \quad \frac{\gcd(ab)c}{b} \in \mathbb{Z}$$

hence f_3 is an integer. Thus $f_0, f_1, f_2, f_3 \in \mathbb{Z}$ and the integrality of the whole sequence (f_n) follows immediately from 5.1.

As to uniqueness of factorization of a primitive lens sequence, assume *a contrario* that two integer quadruples, $\{f_0, f_1, f_2, f_3\}$ and $\{g_0, g_1, g_2, g_3\}$ are the initial terms of two different factorizations of (b_n). Then $g_0/f_0 = p/q$ for some mutually prime $p, q \in \mathbb{Z}$. At least one of p and q is not 1; assume that it is $p \neq 1$. Since $g_i g_{i+1} = f_i f_{i+1}$, we must have

$$\{g_0, g_1, g_2, g_3\} = \{\frac{p}{q}\, f_0, \frac{q}{p}\, f_1, \frac{p}{q}\, f_2, \frac{q}{p}\, f_3\} \ \subset \mathbb{Z}\,.$$

Thus $q|f_0$ and $q|f_2$ (because $\gcd(p,q) = 1$). But this means that $q \mid a$ (since $a = f_0 f_1$), $q \mid b$ (since $b = f_1 f_2$), and $q \mid c$ (since $c = f_2 f_3$), against the assumption of primitivity of the lens sequence. □

The 3-term recurrence 5.1 for the underground sequence involves only α. Another interesting non-linear 4-term recurrence involves only β:

Proposition 5.3. Any underground sequence $\{f_i\}$ of a lens sequence $\{b_i\}$ satisfies the following quadratic recurrence formula:

$$\det \begin{bmatrix} f_n & f_{n+1} \\ f_{n+2} & f_{n+3} \end{bmatrix} \equiv f_{n+3} f_n - f_{n+1} f_{n+2} = -\beta. \tag{5.2}$$

Proof. For any n we have

$$\begin{aligned} f_n f_{n+3} - f_{n+1} f_{n+2} &= \frac{f_n f_{n+1} f_{n+2} f_{n+3}}{f_{n+1} f_{n+2}} - f_{n+1} f_{n+2} \\ &= \frac{b_{n+1} b_{n+3}}{b_{n+2}} - b_{n+2} = \frac{b_{n+1} b_{n+3} - b_{n+2}^2}{b_{n+2}} = -\beta \end{aligned}$$

□

Note that not any initial quadruple (f_1, f_2, f_3, f_4) leads via recurrence 5.1 or 5.6to a sequence that underlines a lens sequence. When do they? First, we notice that the underground sequences of lens sequences have an interesting anatomy. It turns out that they are determined by three-term linear recurrences with *variable* "constants". Here is the central theorem for the underground sequences:

Theorem 5.4 (Underground Sequence Structure)**.** *(i) Let $k, s \in \mathbb{Z}$ be two constants. Define a sequence f by*

$$f_n = \begin{cases} k f_{n-1} - f_{n-2} & \text{if } n \text{ is even} \\ s f_{n-1} - f_{n-2} & \text{if } n \text{ is odd.} \end{cases} \tag{5.3}$$

with some arbitrary initial terms $f_0, f_1 \in \mathbb{Z}$. Define $b_n = f_{n-1} f_n$. Then (b_n) is a lens sequence. The constants of its recurrence formula

$$b_n = \alpha\, b_{n-1} - b_{n-2} + \beta \tag{5.4}$$

are

$$\begin{cases} \alpha = ks - 2 \\ \beta = kf_1^2 + sf_0^2 - ks\, f_0 f_1\,, \end{cases} \tag{5.5}$$

(ii) Every lens sequence is of such type. In particular, for a primitive lens sequence with a seed $(b_{-1}, b_0, b_1) = (a, b, c)$ the underground sequence is defined

$$\begin{aligned} f_0 &= \gcd(a,b), & f_1 &= \gcd(b,c), \\ s &= \frac{a+b}{f_0^2}, & k &= \frac{b+c}{f_1^2}. \end{aligned}$$

Proof. We start with part (ii). Let (f_i) be a sequence defined by 5.3. First, we shall show that the following expression

$$\Delta_n = \det \begin{bmatrix} f_{n-3} & f_{n-2} \\ f_{n-1} & f_n \end{bmatrix} \tag{5.6}$$

is an invariant of a sequence (5.3), that is it does not depend on n. Indeed, let p denote k or s, depending on whether n is even or odd (it will not matter!). Then

$$\begin{aligned} \Delta_n = \det \begin{bmatrix} f_{n-3} & f_{n-2} \\ f_{n-1} & f_n \end{bmatrix} &= \det \begin{bmatrix} f_{n-3} & pf_{n-3} - f_{n-4} \\ f_{n-1} & pf_{n-1} - f_{n-2} \end{bmatrix} \\ = \det \begin{bmatrix} f_{n-3} & -f_{n-4} \\ f_{n-1} & -f_{n-2} \end{bmatrix} &= \det \begin{bmatrix} f_{n-2} & f_{n-1} \\ f_{n-4} & f_{n-3} \end{bmatrix} = \Delta_{n-1}\,. \end{aligned}$$

Thus, by induction, the value of Δ_n does not depend on n. Hence it may be brought down to the first two terms of the sequence and, after simple substitution, shown to be

$$\Delta_n = kf_1^2 + sf_0^2 - ks\, f_0 f_1\,.$$

As to the recurrence formula, calculate the sum $b_{n-1} + b_{n+1}$:

$$\begin{aligned} b_{n+1} + b_{n-1} &= f_n f_{n+1} + f_{n-2} f_{n-1} = (af_{n-1} - f_{n-2})(bf_n - f_{n-1}) + f_{n-2} f_{n-1} \\ &= ks\, f_{n-1} f_n - kf_{n-1}^2 - sf_n f_{n-2} + f_{n-1} f_{n-2} + f_{n-1} f_{n-2} \\ &= ks\, f_{n-1} f_n - f_{n-1}(kf_{n-1} - f_{n-2}) - f_{n-2}(sf_n - f_{n-1}) \\ &= ks\, f_{n-1} f_n - f_{n-1} f_n - f_{n-2} f_{n+1} \\ &= ks\, f_{n-1} f_n - 2f_{n-1} f_n + f_{n-1} f_n - f_{n-2} f_{n+1} \\ &= (ks-2)\, f_{n-1} f_n + (f_{n-1} f_n - f_{n-2} f_{n+1}) \\ &= (ks-2)\, b_n - \Delta_n\,. \end{aligned} \tag{5.7}$$

Now, rename $ks - 2 = \alpha$, and $\Delta_n = -\beta$ (as it does not depend on n). Then (5.7) becomes an inhomogeneous three-term recurrence formula

$$b_{n+1} + b_{n-1} = \alpha\, b_n + \beta\,.$$

Finally, we need to show that (b_n) is actually a lens system sequence. As for α, we need simply to show

$$\alpha = \frac{b_{n-1}}{b_n} + \frac{b_{n+2}}{b_{n+1}}$$

(see Proposition 3.1). Consider the right hand side:

$$\frac{b_{n-1}}{b_n} + \frac{b_{n+2}}{b_{n+1}} = \frac{f_{n-2}f_{n-1}}{f_{n-1}f_n} + \frac{f_{n+1}f_{n+2}}{f_n f_{n+1}} = \frac{f_{n-2}}{f_n} + \frac{f_{n+2}}{f_n} = \frac{f_{n-2}+f_{n+2}}{f_n} = \alpha\,,$$

where the last step is true because of (5.3). Indeed, assume n is even:

$$\begin{aligned}\frac{f_{n-2}+f_{n+2}}{f_n} &= \frac{f_{n-2}+kf_{n+1}-f_n}{f_n}\\ &= \frac{f_{n-2}+k(sf_n-f_{n-1})-f_n}{f_n}\\ &= \frac{ksf_n-kf_{n-1}-f_{n-2}-f_n}{f_n}\\ &= \frac{ksf_n-f_n-f_n}{f_n}\\ &= ks-2=\alpha\,.\end{aligned}$$

As to the formula for β in terms of a seed, follow similar calculations as in the proof of Proposition 5.3:

$$\Delta_n = \det\begin{bmatrix} f_0 & f_1\\ f_2 & f_3\end{bmatrix} = f_0f_3 - f_1f_2 = \frac{f_0f_1f_2f_3-(f_1f_2)^2}{f_1f_2} = \frac{b_1b_3-b_2^2}{b_2} = -\beta\,,$$

This ends the proof of (i). The proof of (ii) follows easily. Let (a,b,c) be a primitive seed of an integer sequence. Define a quadruple of numbers (f_0,f_1,f_2,f_3,f_4) by setting $f_1=\gcd(a,b)$, $f_2=\gcd(b,c)$, and $f_0=a/f_1$, and $f_4=c/f_2$. Then $k=(f_0+f_2)/f_1$ and $s=(f_1+f_3)/f_2$. □

Remark 5.5. Note that Δ_i may be viewed as a quadratic form given by the matrix

$$G = \begin{bmatrix} k & ks\\ 0 & s\end{bmatrix}$$

evaluated on the vector $\mathbf{v}=[f_0,f_1]^T$, i.e., $\Delta_n=\mathbf{v}^TG\mathbf{v}$. In particular, vectors $(f_i,f_{i+1})\in\mathbb{R}$ all stay on a quadratic defined by G.

As a corollary of the above theorem, we arrive at a simple criterion on whether a triple of integers is a good candidate for an integer lens sequence, improving that of Proposition 4.2:

Theorem 5.6 (Integrality Criterion 2)**.** *A triple $(a,b,c)\subset\mathbb{Z}$ is a seed of an integer lens sequence iff*

$$\begin{aligned}(i)&\;\; \gcd(a,b)\gcd(b,c) = b\gcd(a,b,c)\\ (ii)&\;\; (\gcd(a,b))^2 \;\text{ divides }\; (a+b)\gcd(a,b,c)\\ (iii)&\;\; (\gcd(b,c))^2 \;\text{ divides }\; (b+c)\gcd(a,b,c)\end{aligned}$$

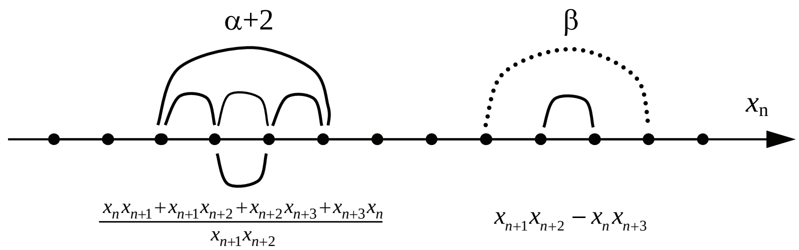


Figure 5.1: Diagrammatic representation of the lens sequence invariants, calculated from the underground sequence (cf. Fig. 3.2)

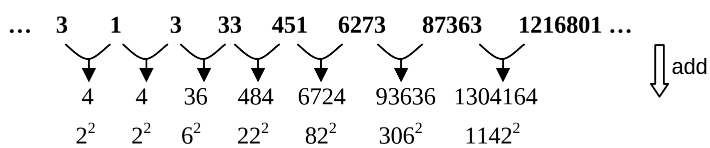
The above result leads to another property of lens sequences:

Corollary 5.7. *The sum of any pair of consecutive terms of a lens sequence is a multiple of a square, namely:*

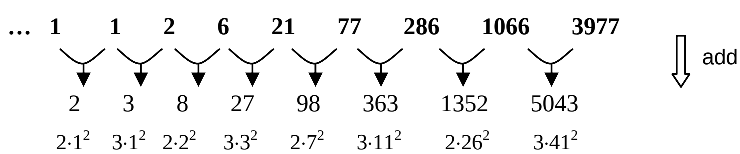
$$b_n + b_{n+1} = \begin{cases} kf_n^2 & \text{if } n \text{ is even} \\ sf_n^2 & \text{if } n \text{ is odd} \end{cases} = \begin{cases} k \cdot \textit{square} & \text{if } n \text{ is even} \\ s \cdot \textit{square} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Elementary: $b_n + b_{n+1} = f_n f_{n-1} + f_{n+1} f_n = f_n(f_{n+1} + f_{n-1}) = f_n p f_n$, where p stands for k or s, depending on the parity of n. □

For example, the sums of two consecutive entries of A011922 are perfect squares:



The sequence A101265 (of label [1,1;3]) gives:



Generating lens sequences

Let us return to the question of generating integer lens sequences. In Section 2, we considered triplets of the form $(a, b; k)$ that label a large family of lens sequences (see (3.1)), but not all of them. The existence of underground sequences allows one to label all lens sequences.

Definition 5.8. A **symbol** of a lens sequence is the quadruple ${}^s(p, q)^k$ which defines the underground sequence (f_i) with $f_0 = p$, $f_1 = q$, and with constant s and k in (5.3), and therefore defines the corresponding lens sequence (b_i), namely, $b_i = f_{i-1} f_i$. More directly, symbol ${}^s(p, q)^k$ defines a lens sequence via its seed $(a, b, c) = ((sp - q)p, pq, q(kq - p))$.

Any integer quadruple ${}^{s}(p,q)^{k}$ leads to an integer lens sequence. And vice versa, given a seed of a primitive sequence (a,b,c), we easily reproduce the symbol:

$$
\begin{aligned}
p &= \gcd(a,b), \qquad & k &= \frac{b+c}{q^2}, \\
q &= \gcd(b,c), \qquad & s &= \frac{a+b}{p^2}.
\end{aligned}
$$

Proposition 5.9. The lens sequence generated by ${}^{s}(p,q)^{k}$ is primitive if and only if

$$\gcd(p,q) = \gcd(p,k) = \gcd(s,q) = 1. \tag{5.8}$$

Proof. Write the "central" four terms of the underground sequences and the corresponding lens sequence:

$$
\begin{aligned}
(f_i): &\qquad \ldots, f_{-1} = (sp-q), f_0 = p, f_1 = q, f_2 = (kq-p), \ldots \\
(b_i): &\qquad \ldots, a = (sp-q)p, b = pq, c = q(kq-p), \ldots
\end{aligned}
$$

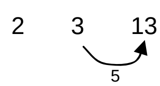
For (b_i) to be primitive we must have $\gcd(a,b,c) = 1$, which implies (5.8). □

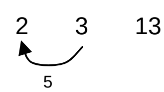
Corollary 5.10. *If the underground sequence (f_i) contains $p = \pm 1$, then the corresponding lens sequence (b_i) admits label $(a,b;k)$. More precisely:*

$$
\begin{gathered}
(a,b;k) \text{ corresponds to the symbol } {}^{a+b}(1,b)^{k}\,; \\
\text{the symbol } {}^{s}(1,b)^{k} \text{ corresponds to } (s-b,b;k)\,.
\end{gathered}
$$

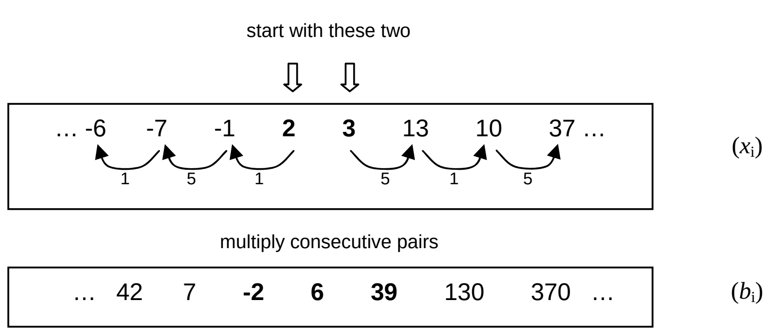
Remark 5.11. To use this generator of sequences as a unique *label* system for lens sequences, one would have to remove the ambiguity of the choice of the initial terms. We may demand that, say, $p = f_0$ has the smallest absolute value among (f_i) and that $|f_{-1}| > f_0 \leqslant f_1$.

Remark on diagrammatic use of symbols. The first of the following two diagrams means that 13 is obtained as $13 = 5 \times 3 - 2$. The second represents equation $2 = 5 \times 3 - 2$:

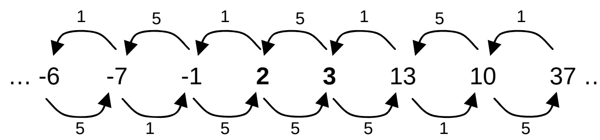




Now, the symbol ${}^{1}(2,3)^{5}$ may be graphically developed into an underground sequence (f_i), and consequently into a lens sequence (b_i), in the following way:



Note that the four central terms of (f_i) suffice to generate the three central terms of (b_i), which yield the constants α and β. The recurrence formula for sequence (f_i) is bilateral and may be represented diagrammatically as shown below:



Example 1 (Vesica Piscis): $\mathbf{b} =$ A011922 $\mathbf{f} =$ A001835 = A079935
$\{b_i\} = (..., 1, 3, 33, 451, 6273, 87363, \ldots)$, $b_{n+1} = 14b_n - b_{n-1} - 8$
$\{f_i\} = (..., 1, 3, 11, 41, 153, 571, \ldots)$, ${}^4(1,1)^4$ (number of domino packings in a $(3 \times 2n)$ rectangle)

Example 2 (Golden Vesica): $\mathbf{b} =$ A064170 $\mathbf{f} =$ A001519
$\{b_i\} = (..., 1, 2, 10, 65, 442, 30, 26, 20737, \ldots)$ $b_{n+1} = 7b_n - b_{n-1} - 3$
$\{f_i\} = (..., 1, 2, 5, 13, 34, 89, 233, \ldots)$, ${}^3(1,1)^3$ (odd Fibonacci numbers)

Example 3: $\mathbf{b} =$ A000466 $\mathbf{f} =$ A005408
$\{b_i\} = (..., -1, 3, 15, 35, 63, 99, 143, 195, 255, \ldots)$ $b_{n+1} = 2b_n - b_{n-1} + 8$
$\{f_i\} = (... -1, 1, 3, 5, 7, 9, 11, 13, 15, \ldots)$, ${}^2(-1,1)^2$ (odd numbers)

Example 4: $\mathbf{b} =$ A081078 $\mathbf{f} =$ A002878
$\{b_i\} = (..., -1, 4, 44, 319, 2204, \ldots)$ $b_{n+1} = 7b_n - b_{n-1} + 15$
$\{f_i\} = (... -1, 1, 4, 11, 29, 76, 199, 521, 1364 \ldots)$, ${}^3(-1,1)^3$ (odd Lucas numbers)

Example 5: $\mathbf{b} =$ A005247 $\mathbf{f} =$ A005247
$\{b_i\} = (..., 2, 2, 3, 6, 14, 35, 90, 234, 611, 1598, \ldots)$ $b_{n+1} = 3b_n - b_{n-1} - 1$
$\{f_i\} = (..., 2, 1, 3, 2, 7, 5, 18, 13, 47, 34, \ldots)$, , ${}^1(2,1)^5$
(even Lucas numbers interlaced with odd Fibonacci numbers)

Example 6: $\mathbf{b} =$ A027941 $\mathbf{f} =$ A005013
$\{b_i\} = (..., 0, 1, 4, 12, 33, 88, 232, 609, \ldots)$ $b_{n+1} = 3b_n - b_{n-1} + 1$
$\{f_i\} = (..., 0, 1, 1, 4, 3, 11, 8, 29, 21, 76, \ldots)$, , ${}^1(1,1)^5$
(odd Lucas numbers interlaced with even Fibonacci numbers)

Example 7: $\mathbf{b} =$A000217 $\mathbf{f} =$ A026741
$\{b_i\} = (..., 0, 1, 3, 6, 10, 15, 21, 28 \ldots)$ $b_{n+1} = 2b_n - b_{n-1} + 1$
$\{f_i\} = (..., 0, 1, 1, 3, 2, 5, 3, 7, 4, 9, \ldots)$, ${}^4(0,1)^1$
(the sequence of natural numbers interlaced with odd natural numbers): $f_n = \begin{cases} n & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even} \end{cases}$

Table 5.1: Examples of underground sequences $\mathbf{f}$ for some integer lens sequences $\mathbf{b}$.

Towards the meaning of the underground sequence

Finally, let us consider yet another recurrence formula for lens sequences and its surprising context. Let us start with this:

Proposition 5.12. Lens sequences obey the following nonlinear 4-step recurrence formula:

$$b_{n+2} = \frac{(b_{n+1} - \beta)(b_n - \beta)}{b_{n-1}} \,. \tag{5.9}$$

Proof. Rewrite the definition of β in the form $ac = b(b-\beta)$. Since the three consecutive terms (a, b, c) may start with any entry of the sequence, let us write its two instances:

$$\begin{cases} b_{n+1}b_{n-1} = b_n(b_n - \beta) \\ b_{n+2}b_n = b_{n+1}(b_{n+1} - \beta) \end{cases}$$

Multiply side-wise to get $b_{n+1}b_{n-1}b_{n+2}b_n = (b_n - \beta)(b_{n+1} - \beta)b_n b_{n+1}$. Canceling the repeated terms results in

$$b_{n-1}b_{n+2} = (b_n - \beta)(b_{n+1} - \beta),$$

which is equivalent to (5.9). □

Corollary 5.13. *A lens sequence satisfies the following identity:*

$$\det \begin{bmatrix} b_{n+1} - \beta & b_{n+2} \\ b_{n-1} & b_n - \beta \end{bmatrix} = 0\,.$$

This leads to yet another implication. The above determinant (5.9) may be written in the form:

$$\det\left(\begin{bmatrix} b_{n+1} & b_{n+2} \\ b_{n-1} & b_n \end{bmatrix} - \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix} \right) = 0 \quad ,$$

which looks like a characteristic equation with β playing the role of the eigenvalue. Note that it does not depend on n. What are the corresponding eigenvectors?

Theorem 5.14. *If (f_i) is the underground sequence of a lens sequence (b_i), then the following "eigen-equation" holds:*

$$\begin{bmatrix} b_{n+1} & b_{n+2} \\ b_{n-1} & b_n \end{bmatrix} \begin{bmatrix} f_{n+1} \\ -f_{n-1} \end{bmatrix} \quad = \quad \beta \begin{bmatrix} f_{n+1} \\ -f_{n-1} \end{bmatrix}.$$

Proof. Write b's in terms of f's and use the definition of β written in terms of f's. □

6 Summary

A lens sequence is an integer sequence (b_i) that satisfies two conditions:

$$\begin{array}{lll} \text{(i)} & b_n = \alpha b_{n-1} - b_{n-2} + \beta & \text{[recurrence formula]} \\ \text{(ii)} & a^2 + b^2 = \alpha\, ab + \beta(a+b) & \text{[compatibility relation]} \end{array}$$

where α and β are constants and a and b are any two consecutive terms of (b_i). These two conditions assure that the sequence has a geometric realization in terms of the curvatures of a chain of circles inscribed in a symmetric lens (the space of the overlap of the interiors or exteriors of two congruent circles). The sequence constants may be viewed as invariants of a process $i \longrightarrow b_i$. They may be calculated from a *seed*, i.e., any three consecutive sequence terms (a, b, c):

$$\alpha = \frac{ab + bc + ca}{b^2} - 1 \text{ and } \beta = \frac{b^2 - ac}{b}$$

or, for α, alternatively

$$\alpha = \frac{b_{n-1}}{b_n} + \frac{b_{n+2}}{b_{n+1}}\,.$$

Other, nonlinear, recurrence formulas for the lens sequence include:

[two-step formula]	$2b_{n+1} = b_n\alpha + \beta \pm \sqrt{(\alpha^2-4)\,b_n^2 + 2(\alpha+2)\beta\,b_n + \beta^2}$	
[three-step formula]	$b_{n+1}b_n + b_{n+1}b_{n-1} + b_nb_{n-1} = (\alpha+1)b_n^2$	(only α)
[three-step formula]	$b_nb_n - b_{n+1}b_{n-1} = \beta b_n$	(only β)
[four-step formula]	$b_{n+2}b_{n-1} = (b_{n+1}-\beta)(b_n-\beta)$	
[four-step formula]	$b_{n+1}b_{n-1} + b_nb_{n-2} = \alpha b_nb_{n-1}$	

The sequence constants have a geometric meaning: α codes the angle under which the circles forming the lens intersect (if they do), or, more generally, the Pedoe product of the lens circles. The value of β reflects the size of the system. There are two basic properties determined by geometry: (a) the sum of the inverses is determined by the length of the lens, and (b) the limit of the ratio of consecutive terms is determined by the aforementioned lens angle:

$$\sum_n \frac{2}{b_n} = \frac{\sqrt{\alpha^2-4}}{-\beta} \qquad \text{and} \qquad \lim_{i\to\infty}\frac{b_{i+1}}{b_i} = \frac{\alpha+\sqrt{\alpha^2-4}}{2} = \lambda\,,$$

The number λ, the *characteristic constant* of (b_i), allows one to express the lens sequence by a Binet-type formula

$$b_n = w\lambda^n + \bar{w}\bar{\lambda}^n + \gamma$$

where

$$w = \frac{a-2b+c}{2(\alpha-2)} + \frac{c-a}{2(\alpha^2-4)}\sqrt{\alpha^2-4}, \quad \gamma = \frac{-\beta}{\alpha-2}\,,$$

and where the bar denotes natural conjugation in the field $\mathbb{Q}(\sqrt{\alpha^2-4})$. The constant λ is an example of a (quadratic) Pisot number, an algebraic integer, the powers of which approximate natural numbers. In particular, $b_n \approx w\lambda^n + \gamma$. Lens sequences can be expressed also as combinations of Chebyshev polynomials.

The most mysterious property of a lens sequence is that its terms may be formed by taking products of pairs of consecutive terms of another sequence. This "underground" sequence has an alternating recurrence rule, different for odd and even terms. Namely $b_n = f_{n-1}f_n$, where:

$$f_n = \begin{cases} kf_{n-1} - f_{n-2} & \text{if } n \text{ is even} \\ sf_{n-1} - f_{n-2} & \text{if } n \text{ is odd.} \end{cases}$$

The constants of the sequence (b_i) may now be expressed as

$$\begin{cases} \alpha = ks - 2 \\ \beta = sf_0^2 + kf_1^2 - ks\,f_0f_1\,. \end{cases}$$

It follows that the integer lens sequences may be determined by four arbitrary integers

$$^{s}(f_0, f_1)^{k}\,.$$

Choosing for f_0 the term with smallest absolute value allows one to treat the above quadruple as a **symbol** that labels the corresponding lens sequence.

The underground sequences automatically satisfy the following two recurrence formulas:

$$(i)\ f_{n+2} + f_{n-2} = \alpha f_n$$

$$(ii)\ f_{n+2}f_{n-1} - f_n f_{n+1} = -\beta \qquad \text{or} \qquad \det \begin{bmatrix} f_{n-1} & f_n \\ f_{n+1} & f_{n+2} \end{bmatrix} = -\beta$$

More precisely, the set of the sequences that are underground sequences for lens sequences coincides with the intersection $\Lambda \cap \Delta$, where Λ denotes the space of sequences satisfying linear recurrence (i), and Δ denotes the set of sequences satisfying (ii).

An intriguing property holds — the eigenvectors of matrices assembled from the terms of a lens sequence are vectors with entries from the corresponding underground sequence:

$$\begin{bmatrix} b_{n+1} & b_{n+2} \\ b_{n-1} & b_n \end{bmatrix} \begin{bmatrix} f_{n+1} \\ -f_{n-1} \end{bmatrix} = \beta \begin{bmatrix} f_{n+1} \\ -f_{n-1} \end{bmatrix}.$$

But the full meaning of the underground sequences remains to be understood.

Appendix

Below, we summarize the general formulas for each of the five types of symmetric integer lens sequences. Recall that L = length of the lens, R = radius of the lens circles, δ = their relative distance, and λ = characteristic constant. Due to symmetry, the sums of the reciprocals are curtailed to one (right) tail of the sequence.

---

1. Seed: $[n, 1, n]$. Symbol: $^{n+1}(1,1)^{n+1}$. Recurrence: $\alpha = (n+1)^2 - 2,\ \beta = 1 - n^2$

   Geometry: $R = \frac{n+1}{n-1}$, $L = 2\frac{n+3}{n-1}$, $\delta = \frac{4}{n+1}$. (Inner chain)

   Characteristic constant: $\lambda = \left(\frac{n+1+\sqrt{(n+3)(n-1)}}{2}\right)^2 = \frac{n^2+2n-1+(n+1)\sqrt{(n+3)(n-1)}}{2}$

   Binet: $b_k = \frac{\lambda^k + \bar{\lambda}^k + n + 1}{n+3}$

   Sum of reciprocals: $\sum_{k=0}^{\infty} 1/b_k = 1 + \frac{1}{n} + \ldots = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{n+3}{n-1}}$

---

2. Seed: $[n, 1, 1, n]$. Symbol: ${}^2(1, 1)^{n+1}$. Recurrence: $\alpha = 2n$, $\beta = 1 - n$.

   Geometry: $R = 2\frac{n+1}{n-1}$, $L = 2\sqrt{\frac{n+1}{n-1}}$, $\delta = \frac{4}{\sqrt{2(n+1)}}$. (Inner chain)

   Characteristic constant: $\lambda = \left(\frac{\sqrt{2n+2}+\sqrt{2n-2}}{2}\right)^2 = \frac{n+\sqrt{n^2-1}}{2}$

   Binet: $b_k = \frac{1}{4}\left(1 + \sqrt{\frac{n-1}{n+1}}\right)\lambda^k + \frac{1}{4}\left(1 - \sqrt{\frac{n-1}{n+1}}\right)\lambda^{-k} + \frac{1}{2}$

   Sum of reciprocals: $\sum_{k=1}^{\infty} 1/b_k = 1 + \frac{1}{n} + \ldots = \sqrt{\frac{n+1}{n-1}}$

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3. Seed: $[n, 2, 2, n]$. Symbol: ${}^1(2, 1)^{n+2}$. Recurrence: $\alpha = n$, $\beta = 2 - n$.

   Geometry: $R = \frac{n+2}{n-2}$, $L = 2\sqrt{R} = 2\frac{n+2}{n-2}$, $\delta = \frac{4}{\sqrt{n+2}}$. (Inner chain)

   Characteristic constant: $\lambda = \left(\frac{\sqrt{n+2}+\sqrt{(n-2)}}{2}\right)^2 = \frac{n+\sqrt{n^2-4}}{2}$

   Binet: $b_k = \frac{1}{2}\left(1 + \sqrt{\frac{n-2}{n+2}}\right)\lambda^k + \frac{1}{2}\left(1 - \sqrt{\frac{n-2}{n+2}}\right)\lambda^{-k} + 1$

   Sum of reciprocals: $\sum_{k=1}^{\infty} 1/b_k = \frac{1}{2} + \frac{1}{n} + \ldots = \frac{1}{2}\sqrt{\frac{n+2}{n-2}}$

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4. Seed: $[n, -1, n]$. Symbol: ${}^{n-1}(1, 1)^{n+1}$. Recurrence: $\alpha = (n-1)^2 - 2$, $\beta = n^2 - 1$.

   Geometry: $R = -\frac{n-1}{n+1}$, $L = 2\frac{n-3}{n+1}$, $\delta = \frac{4}{n-1}$. (Outer chain)

   Characteristic constant: $\lambda = \left(\frac{n-1+\sqrt{(n-3)(n+1)}}{2}\right)^2 = \frac{n^2-2n-1+(n-1)\sqrt{(n-3)(n+1)}}{2}$

   Binet: $b_k = \frac{\lambda^k + \bar{\lambda}^k - (n-1)}{n-3}$

   Sum of reciprocals: $\sum_{k=1}^{\infty} 1/b_k = \frac{1}{n} + \ldots = \frac{4}{n+1}$

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5. Seed: $[0, 1, n]$. Symbol: ${}^1(1, 1)^{n+1}$. Recurrence: $\alpha = n - 1$, $\beta = 1$.

   Geometry: $R = -(n + 1)$, $L = 2\sqrt{(n + 1)(n - 3)}$, $\delta = \frac{4}{\sqrt{n+1}}$. (Outer chain)

   Characteristic constant: $\lambda = \left(\frac{n+1+\sqrt{(n-3)}}{2}\right)^2 = \frac{n-1+\sqrt{(n+1)(n-3)}}{2}$

   Binet: $b_k = \frac{1}{2}\left(\frac{n-2}{n-3} + \frac{n}{\sqrt{(n+1)(n-3)}}\right)\lambda^k + \frac{1}{2}\left(\frac{n-2}{n-3} - \frac{n}{\sqrt{(n+1)(n-3)}}\right)\lambda^{-k} + \frac{1}{n-3}$

   Sum of reciprocals: $\sum_{k=0}^{\infty} 1/b_k = 1 + \frac{1}{n} + \ldots = \frac{n+1-\sqrt{(n+1)(n-3)}}{2}$

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