ON A DIOPHANTINE EQUATION THAT GENERATES ALL APOLLONIAN GASKETS

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Abstract. A remarkably simple Diophantine quadratic equation is known to generate all Apollonian integral gaskets (disk packings). A new derivation of this formula is presented here based on inversive geometry. Also occurrences of Pythagorean triples in such gaskets is discussed.

1. Introduction

Apollonian disk packing (or Apollonian gasket) is a pattern obtained by starting with three mutually tangent circles of which one contains the other two, then recursively inscribing new circles (disks) in the curvilinear triangular regions (called "ideal triangles") formed between the circles. Figure 1 shows a few examples, including (a) a special case of the noncompact "Apollonian Strip", (b) the Apollonian Window which is the only case that has symmetry D_2, (c) the regular threefold gasket, which has symmetry D_3, and (d) a general gasket that may have no mirror symmetry.

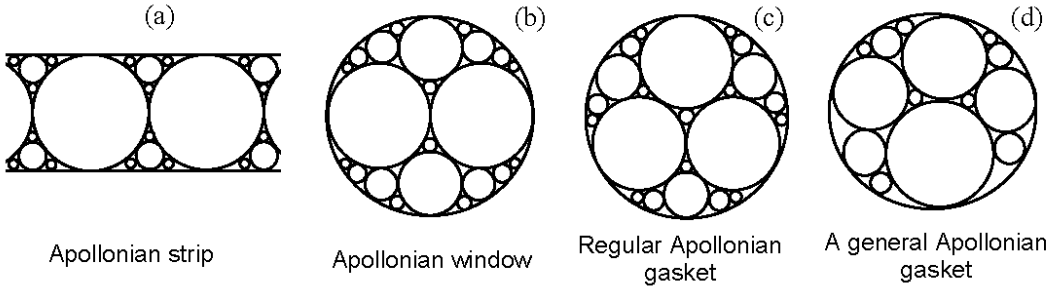

Figure 1. Various Apollonian disk packings

It turns out that there exist an infinite number of *integral* Apollonian packings, where the curvature of every circle/disk is an integer.

Recall that four pairwise tangent circles are said to be in *Descartes configuration*. Suppose a, b, c, d are curvatures of four such circles. Then Descartes' circle formula states

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2. \tag{1.1}$$

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One may use this formula to obtain the curvature of the fourth circle d, given the first three. Being a quadratic equation, it yields two solutions, say d_1 and d_2, such that

$$d_1 + d_2 = d(a + b + c). \tag{1.2}$$

Disks d_1 and d_2 will be called *Boyd-dual*. As observed by Boyd [1], this "linearized" version of (1.1) provides a handy tool for determining the curvatures in a particular Apollonian gasket starting with the first four values. In particular, it follows that if the first four values are integers, so are all.

Remark. For consistency, the curvature of the circle that contains all of the remaining ones must be given a negative value for equations (1.1) and (1.2) to hold. It is customary to replace the term "curvature" by "bend" to account for this convention. Another way of looking at this is to think of disks rather than circles, where the greatest circle is the boundary of an exterior, unbounded, region. This way no two disks in an Apollonian gasket overlap.

In this note, Apollonian gaskets will be labeled by the bends of the five greatest circles, i.e., by the five least bends. Why five will become clear when we consider symmetries, Appendix A.

Examples of integral Apollonian gaskets include these:

$(0,\ 0,\ 1,\ 1,\ 1)$	– Apollonian belt (Figure 1a)
$(-1,\ 2,\ 2,\ 3,\ 3)$	– Apollonian window (Figure 1c)
$(-2,\ 3,\ 6,\ 7,\ 7)$	– less regular gasket, but with D_1 symmetry
$(-6,\ 11,\ 14,\ 15,\ 23)$	– quite irregular case

Note that the regular gasket (see Figure 1) cannot have integer bends, as its quintet of curvatures is, up to scale, $(1 - \sqrt{3},\ 2,\ 2,\ 2,\ 10 + 2\sqrt{3})$, hence its curvatures are populated by elements of $\mathbb{Z}[\sqrt{3}]$. Due to (1.2), the integrality of the first four circles determines integrality of all disks in the packing. An integral Apollonian packing is **irreducible** if the bends have no common factor except 1.

The problem is to classify and determine all irreducible integral Apollonian gaskets.

2. Integer disk packing – the formula

All integral Apollonian disk packings may be determined using a simple Diophantine quadratic equation with constraints. The derivation of this formula is a much simpler alternative to that of "super-Apollonian packing" [2]–[4] and is based on inversive geometry.

Theorem 2.1. *There is a one to one correspondence between the irreducible integral Apollonian gaskets and the irreducible quadruples of non-negative integers* $B, k, n, \mu \in \mathbb{N}$ *that are solutions to quadratic equation*

$$B^2 + \mu^2 = kn \tag{2.1}$$

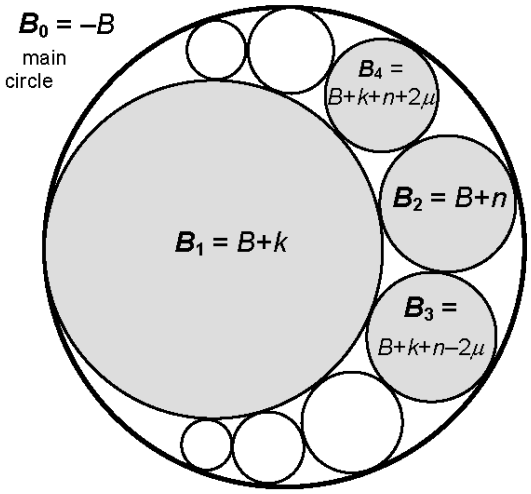


FIGURE 2. An Apollonian gasket and its four greatest circles (smallest curvatures)

with constraints

$$(i) \quad 0 \le \mu \le B/\sqrt{3},$$
$$(ii) \quad 2\mu \le k \le n.$$

Every solution to (2.1) *corresponds to an Apollonian gasket with the following quintet of the major bends (curvatures):*

$$(B_0,\, B_1\,, B_2,\, B_3,\, B_4) = (-B,\, B+k,\, B+n,\, B+k+n-2\mu,\, B+k+n+2\mu)$$

Corollary 1. *The Apollonian gasket is integral iff* $gcd(B, k, n) = 1$.

Figure 2 locates the curvatures of the theorem in the Apollonian gasket. Note that the triple of integers $(B,\ \mu,\ k)$ is also a good candidate for a label that uniquely identifies an Apollonian gasket (since n is determined by: $n = (B^2 + \mu^2)/k$).

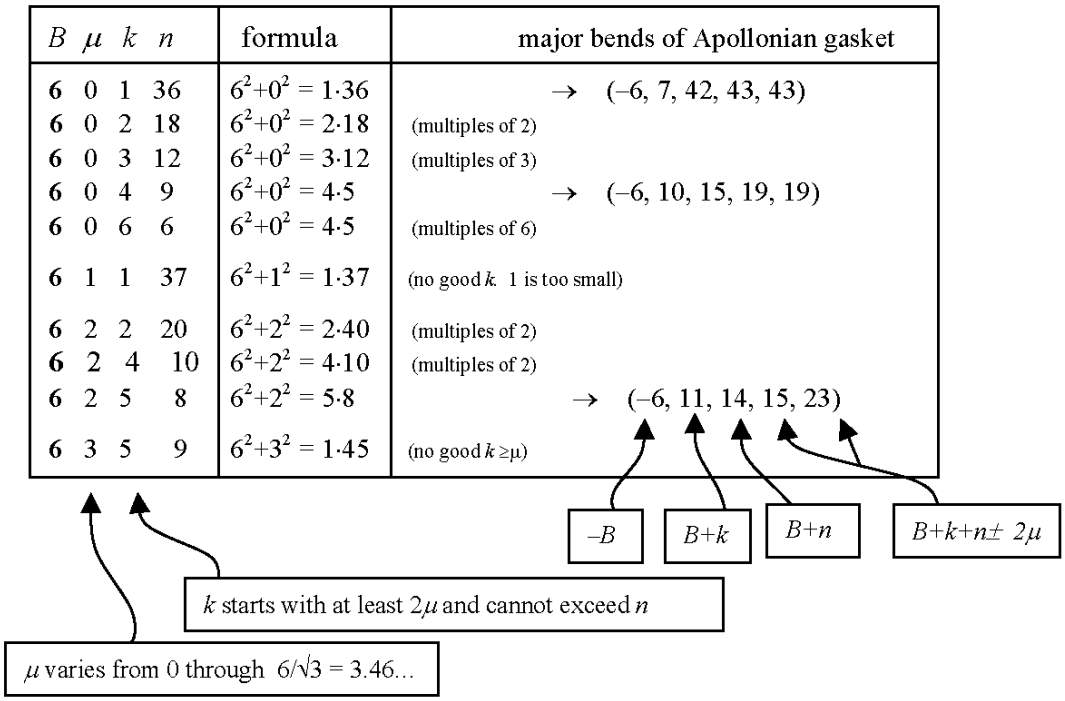
Equation (2.1) leads to an algorithm producing all integral Apollonian gaskets, ordered by the curvatures, presented in Figure 3:

```
for B = 1 to whatever you wish
      for  μ = 0   to   int(B/√3)
            evaluate  H = B² + μ²
            for  k = 2μ  to  int(√H)
                 if  n = H/k is an integer and  gcd(B,k,n) = 1
                 then the quintet of main curvatures is
                  (–B, B+k, B+n, B+k+n–2μ, B+k+n+2μ)
            next k
      next μ
next B
```

FIGURE 3. An algorithm that produces all integral disk packings.

Appendix A shows the first 183 entries for the main bend varying from 0 through 32.

Example 1. The table below shows that there are only 3 Apollonian gaskets for $B = 6$.



Proof Theorem 2.1. (We denote a circle and its curvature by the same symbol). Consider an Apollonian gasket of disks inscribed inside a circle of curvature B (bend equal to $-B$). Draw an axis through the center of this circle and the center of the next largest circle B_1 (the horizontal axis A in Figure 4). Inverting the gasket through a circle K of radius $2/B$ will produce an Apollonian belt, shown in the figure on the right side of the gasket. Denote its width by 2ρ. Lines L_0 and L_1 are the images of B and B_1, respectively.

The next largest circle, B_2, will show up in the strip as a circle B_2'. It will intersect the axis A. Various Apollonian gaskets will result by varying the strip's width (that is the parameter ρ) and the height h of the center of the circle B_2' above the axis. Clearly, the range of h is the interval $[0, \rho)$, as going beyond would result in repetitions of the Apollonian arrangements.

Recall the basic formulas of inversive geometry: inversion through a circle of radius K centered at O carries a point at distance d from O to a point at distance d' according to

$$dd' = K^2 \tag{2.2}$$

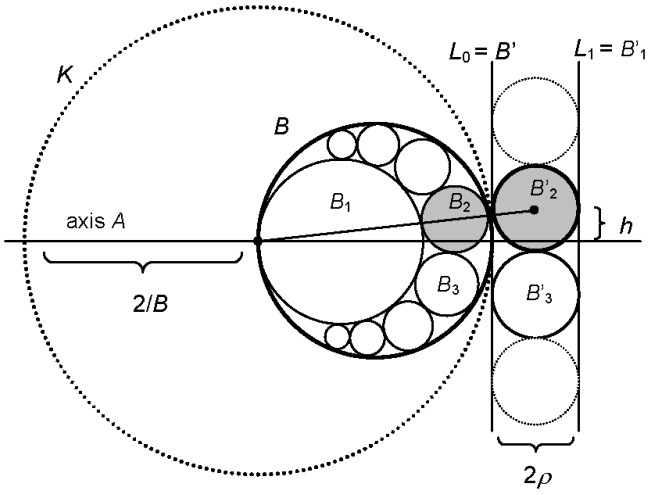


FIGURE 4. The method of inversion.

and a disk of radius r with center distance d from O is mapped to a disk of radius r' with its center located at distance d' according to

$$\text{(i) } r' = \frac{rR^2}{d^2 - r^2}\,, \qquad \text{(ii) } d' = \frac{dR^2}{d^2 - r^2}\,. \tag{2.3}$$

We shall analyze the Apollonian gasket as the inversive image of the Apollonian strip, disk by disk:

Step 0. Recall that $B \in N$ represents the curvature of the encompassing circle of the gasket.

Step 1. To calculate the curvature B_1 of the greatest circle inside (the image of L_1), apply (2.3)i) to the point of intersection of L_1 and axis A to get:

$$B_1 = B + \underbrace{B^2\rho}_{k}\,.$$

Since B is an integer, so is the last term; denote it by $k = B^2\rho$. Clearly, k can be any nonnegative integer (to make $B_1 \geq B$).

Step 2. For B_2, the image of B'_2, use (2.3)ii to get

$$\begin{aligned} B_2 = \frac{1}{r'} = \frac{1}{\rho}\cdot\frac{d^2-\rho^2}{2/B)^2} &= \frac{B^2}{4\rho}\cdot(d^2-\rho^2) && \text{(simplification)} \\ &= \frac{B^2}{4\rho}\cdot\left((2/B+\rho)^2 + h^2 - \rho^2\right) && \text{(Pythagorean thm)} \\ &= B = \underbrace{\frac{4+h^2B^2}{4\rho}}_{n} \end{aligned}$$

As before, we conclude that the last term must be an integer; denote it by n. Clearly, $n \geq k$ (to make $B_2 \geq B_1$).

Step 3. Similarly, we get a formula for the next largest circle B_3 located below B_2, the image of B_3'. Simply use the above formula with $h' = h - 2\rho$ instead of h to get

$$\begin{aligned} B_3 &= \frac{1}{\rho} \cdot \frac{d^2 - \rho^2}{(2/B)^2} = \frac{B^2}{4\rho} \cdot \left((2/B + \rho)^2 + (h^2 - \rho^2)^2 - \rho^2\right) \\ &= B \ + \ \underbrace{\rho B^2}_{k} \ + \ \underbrace{\frac{4 + h^2 B}{4\rho}}_{n} \ - \ \underbrace{hB^2}_{m} \end{aligned}$$

Quite pleasantly, the first three terms coincide with terms from previous steps. Since we have already established that they must be integers, so is the last one; denote it by m. Thus we have three integers defined by the geometry of the construction:

$$n = \frac{4 + h^2 B^2}{4\rho}, \quad k = \rho B^2, \quad m = hB^2. \tag{2.4}$$

Integers k, n and m are not independent; take the definition for n and eliminate h and r from it to get

$$4nk = 4B^2 + m^2,$$

from which it follows immediately that m must be even, say $m = 2\mu$. Reduce the common factor of 4 to get the "master equation" (2.1).

As to the **constraints**, the order of the curvatures gives three inequalities:

$$B_1 \geq B \Rightarrow k \geq 0, \quad B_2 \geq B_1 \Rightarrow n \geq 0, \quad B_3 \geq B_2 \Rightarrow k \geq 2\mu. \tag{2.5}$$

The additional upper bound for μ comes from the fact that k takes its greatest value at $k = \sqrt{B^2 + \mu^2}$. Thus the last inequality of (2.5), $k \geq 2\mu$, implies:

$$\sqrt{B^2 + \mu^2} \geq 2\mu$$

and therefore (after squaring)

$$B^2 > 3\mu^2.$$

This ends the proof. □

3. Symbols of the circles in an Apollonian gasket

The **symbol of a circle** [5, 6] is a formal fraction

$$\frac{\dot{x}, \dot{y}}{b}$$

where $b = 1/r$ denotes the bend (signed curvature) of the circle and the position of the center is

$$(x, y) = \left(\frac{\dot{x}}{b}\, \frac{\dot{y}}{b}\right).$$

By **reduced coordinates** we mean the pair $(\dot{x}, \dot{y})$. In the case of the Apollonian Window (packing with the major curvatures $(-1,\ 2,\ 2,\ 3,\ 3)$), the reduced coordinates and the bend of each circle are integers, see Figure 5.

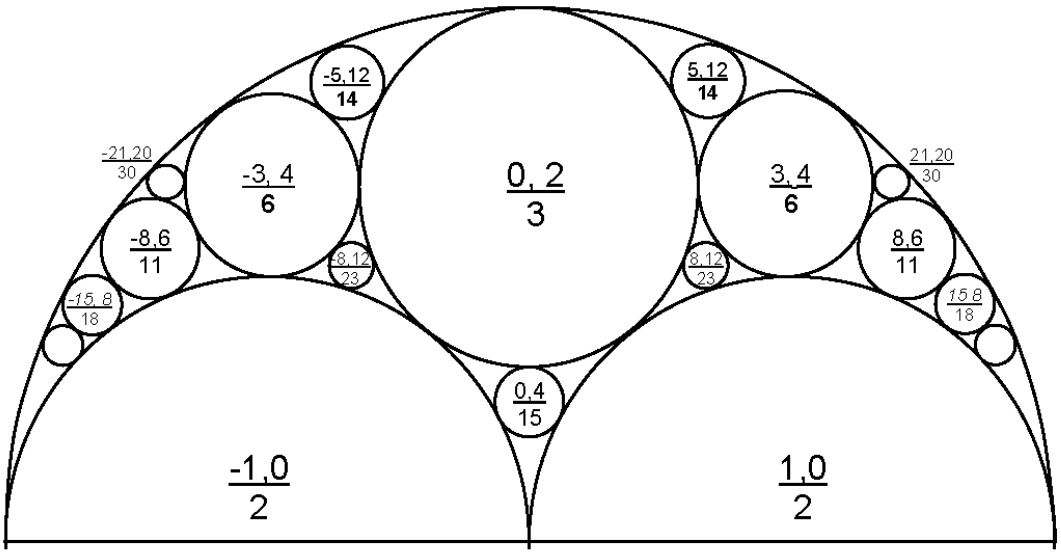


FIGURE 5. Apollonian Window with labels.

Boyd's linearization (1.2) for bends in Descartes configurations holds for the reduced coordinates, since they satisfy a quadratic equation due to the extended Descartes Circle Theorem [2, 3, 4]. Thus:

$$\dot{x}_4 + \dot{x}'_4 = 2(\dot{x}_1 + \dot{x}_2 + \dot{x}_3) \quad \text{and} \quad \dot{y}_4 + \dot{y}'_4 = 2(\dot{y}_1 + \dot{y}_2 + \dot{y}_3). \tag{3.1}$$

Remark. The geometric interpretation of the integrality of the labels. Every circle in the Apollonian Window is an integer multiple of its radius above the horizontal axis and from the vertical axis. This is a generalization of Pappus' Arbelos Theorem [11] for a single chain in an arbelos-like figure, illustrated in Figure 6.

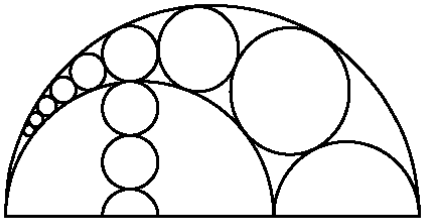


FIGURE 6. Classical Pappus' Arbelos Theorem: the base is divided randomly, yet every circle in the chain is an even multiple of its radius above the axis.

The question is whether the same may be expected for other integer Apollonian disk packings, that is: Are the three numbers in the label all integers?

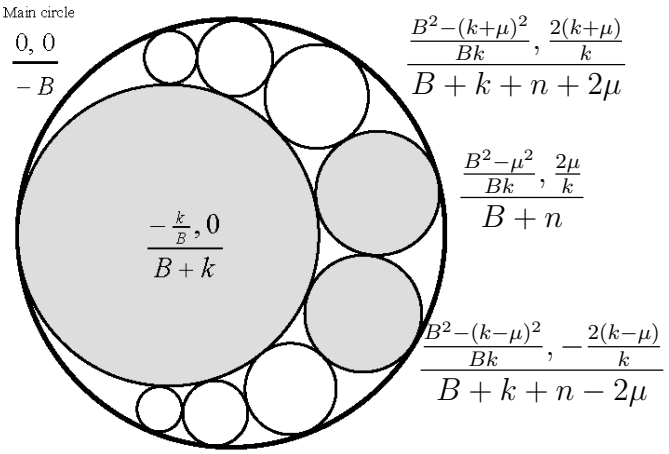


FIGURE 7. Labels of circles in the Apollonian packing $[B,\ \mu,\ n,\ k]$

Proposition 3.1. *In the case of the coordinate system with the center located at the center of circle B (see Figure 4), the labels for the Apollonian gasket generated from (B, n, k, μ) are as presented in Figure 7.*

Proof. Direct calculations. □

One can check that the circles of curvatures $B + k + n \pm \mu$ are Boyd-dual with respect to the triple of mutually tangent circles of bends $(-B,\ B + n,\ B + k)$.

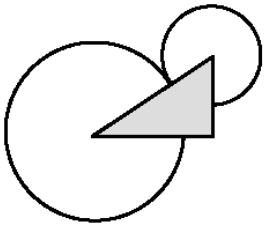
The above proposition implies that the reduced coordinates for circles are fractional unless $B|k$ (see the symbol for B_1 in the figure). Since we have also $2\mu \leq k$ (constraints) and the need for $k|2\mu$ (see the symbols for the reduced y-coordinate), we conclude that we would need $\mu = 0$ or $k = 2\mu$.

This happens in two cases: Apollonian Window $(-1, 2, 2, 3, 3)$, and Apollonian Strip $(0, 0, 1, 1, 1)$, that is for $[B, \mu, k, n]$ equal $[1, 0, 1, 1]$ or $[0, 0, 0, 1]$, respectively. However, if $k|\mu$ (which happens "often", whenever $\mu = 0$), all reduced y-coordinates are integer. This is a special (integral) case of Pappus' observation, since $\mu = 0$ corresponds to the pattern shown in Figure 6, see Appendix).

4. Pythagorean triples in Apollonian gaskets

Given two tangent circles C_1 and C_2, we construct a triangle whose hypotenuse joins the centers and the other sides of which are horizontal or vertical with respect

to some fixed axes.



We shall associate with this triangle a triple of numbers, namely

$$(\Delta, \Gamma, H) = (b_1\dot{x}_2 - b_2\dot{x}_1, b_1\dot{y}_2 - b_2\dot{y}_1, b_1 + b_2). \tag{4.1}$$

The actual size of the triangle's sides is the above triple scaled down by the factor of b_1b_2. Hence the **symbol** for such triangles:

$$\frac{\Delta, \Gamma, H}{b_1 b_2}\,. \tag{4.2}$$

Clearly, $\Delta^2 + \Gamma^2 = H^2$. If the reduced coordinates are integers, then the triples (4.1) are Pythagorean triples! The Apollonian Window thus contains infinitely many Pythagorean triples [5, 6]. A few are displayed in Figure 8.

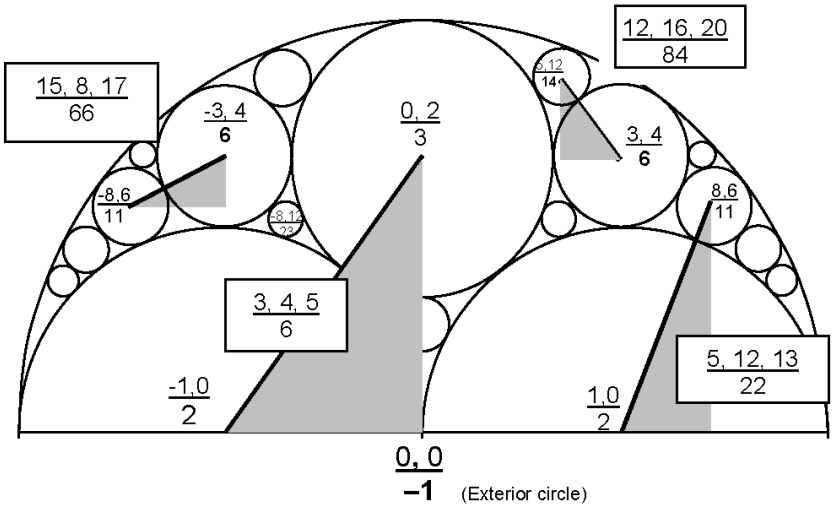


FIGURE 8. Pythagorean triples in the Apollonian Window

Now, for the arbitrary integer packing. Consider the four major circles B_0, B_1, B_2, B_3 in an Apollonian gasket (Figure 6). Pairwise, they determine six right triangles. Each of the thick segments in Figure 9 represents the hypotenuse of one of them. This sextet will be called the **principal frame** for the gasket.

We shall prove later that if the corresponding triples Δ, Γ, H are integers, so are all triples in the gasket, But first:

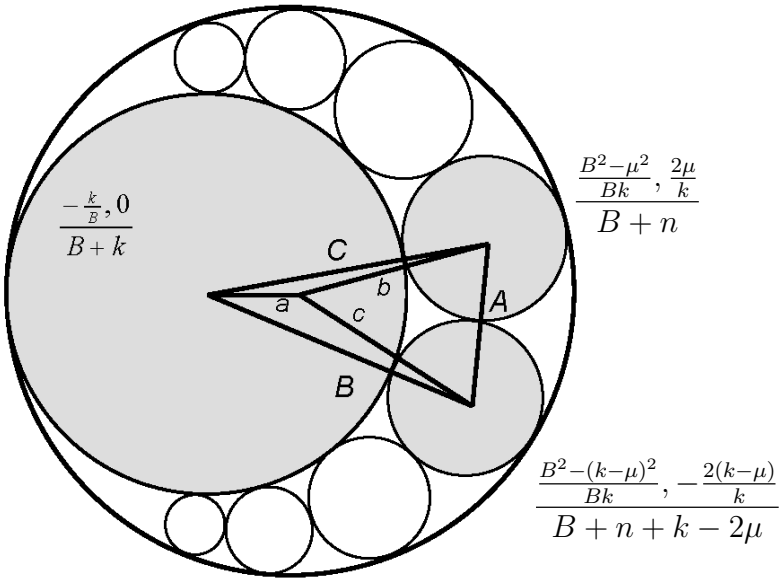


FIGURE 9. Labels of circles in the Apollonian packing $[B,\ \mu,\ n,\ k]$

Proposition 4.1. *The labels of the six right triangles in an Apollonian gasket whose hypotenuses join the pairs of the first four greatest circles are, in notation as in Figure 7 and 9, as follows:*

$$\begin{aligned}
a &= \frac{k,\ 0,\ k}{-B(B+k)} \\
b &= \frac{n-2B^2/k,\ -2\mu B/k,\ n}{-B(B+n)} \\
c &= \frac{k+n-2\mu-2B^2/k,\ 2B-2\mu B/k,\ k+n-2\mu}{-B(B+k+n-2\mu)} \\
A &= \frac{-2b-k+2\mu+4B\mu/k,\ -2(B-n+\mu+2B^2/k),\ 2B+2n+k-2\mu}{(B+n)(B+k+n-2\mu)} \\
B &= \frac{-2B+n-2\mu-2B^2/k,\ 2(B+k-\mu-2B\mu/k),\ 2B+n+2k-2\mu}{-B(B+k+n-2\mu)} \\
C &= \frac{2B+k-n+2B^2/k,\ 2\mu+2\mu B/k,\ 2B+k+n}{(B+k)(B+n)}
\end{aligned} \tag{4.3}$$

Proof. Direct calculation from the data of Proposition 3.1. □

The master equation is used to bring the triples to a form where the only fractional terms contain B/k as a factor. Note that, in general, each of the entries for the reduced coordinates is a linear combination with integer coefficients of B, k, n and μ plus the only, possibly, non-integer term, a multiple of $2B\mu/k$ or $2B^2/k$. The fact that it can be done proves the following theorem:

Theorem 4.1. *If $k|2B^2$ (or, equivalently $k|2B\mu$) then all triples (4.1) in the integer Apollonian gasket (B, μ, k) are integers (form Pythagorean triples).*

Proof. The following implication is direct

$$k|2B^2 \Rightarrow k|(2\mu k - 2\mu^2) \Rightarrow k|2\mu^2.$$

Combining the premise with the result: $k^2|2B^2 2\mu^2 \Rightarrow k^2|4(B\mu)^2 \Rightarrow k|2\mu B$. Every entry of (4.3) may be expressed as a linear integral combination of B, k, n, μ and an extra term of either $2B\mu/k$ or $2B\mu/k$. This proves the claim. □

Integer Pythagorean triples happen frequently. For instance, each of the conditions: $\mu = 0$, or $k = 1$, or $k = 2$, is sufficient.

5. Linear recurrence for Pythagorean triples in an Apollonian gasket

Consider four circles in Descartes configuration, $C_1, \ldots, C_4$ (see Figure 10). The four centers determine six segments that we view as hypotenuses of triangles. They give, after rescaling, six Pythagorean triangles with sides denoted as follows:

$$\begin{array}{rrcll} \text{horizontal side:} & \Delta_{ij} & = & b_i b_j (x_i - x_j) & = \dot{x}_i b_j - \dot{x}_j b_i \\ \text{vertical side:} & \Gamma_{ij} & = & b_i b_j (y_i - y_j) & = \dot{y}_i b_j - \dot{y}_j b_i \\ \text{hypotenuse:} & B_{ij} & = & b_i b_j (1/b_i + 1/b_j) & = b_i + b_j \end{array} \tag{5.1}$$

with $i, j = 1, \ldots 4$. Note that $\Delta_{ij} = -\Delta_{ji}$ and $\Gamma_{ij} = -\Gamma_{ji}$ but $B_{ij} = B_{ji}$. They form the frame of the Descartes configuration.

Now, complement the picture with a circle C_4', Boyd-dual to C_1 (see Figure 10). This leads to a new frame, the frame of the Descartes configuration C_4', C_2, C_3, C_4. Quite interestingly, the elements of the new frame are linear combinations of the elements of the initial frame.

Theorem 5.1. *Following the notation of* (5.1) *and Figure 7, the transition of frames is given by*

$$\begin{aligned} \Delta'_{41} &= -\Delta_{41} + 2\Delta_{21} - 2\Delta_{13} \\ \Gamma'_{41} &= -\Gamma_{41} + 2\Gamma'_{21} - 2\Gamma'_{13} \\ B'_{41} &= -B_{41} + 2B_{12} + 2B_{13} \end{aligned}$$

Proof. Using the linear relations (1.2) and (3.1), we can express the position and the curvature of the new, fifth, circle in terms of the initial four:

$$\begin{aligned} \dot{x}'_4 &= 2\dot{x}_1 + 2\dot{x}_2 + 2\dot{x}_3 - \dot{x}_4 \\ \dot{y}'_4 &= 2\dot{y}_1 + 2\dot{y}_2 + 2\dot{y}_3 - \dot{y}_4 \\ b'_4 &= 2b_1 + 2b_2 + 2b_3 - b_4 \end{aligned}$$

We can calculate the Pythagorean vectors for each pair $C_4'C_i$, $i = 1, 2, 3$. By some magic, due to the following regroupings and adding zeros, these can be expressed in

terms of the initial four Pythagorean vectors only, as shown here for the pair $C_4'C_1$:

$$\begin{aligned}\Delta_{41}' &= \dot{x}_4' b_1 - \dot{x}_1 b_4' \\ &= (2\dot{x}_1 + 2\dot{x}_2 + 2\dot{x}_3 - \dot{x}_4)b_1 - \dot{x}_1(2b_1 + 2b_2 + 2b_3 - b_4) \\ &= \dot{x}_1 b_4 - \dot{x}_4 b_1 + 2(\dot{x}_2 b_1 - \dot{x}_1 b2_1) + 2(\dot{x}_3 b_1 - \dot{x}_1 b_3) \\ &= -Delta_{41} + 2\Delta_{21} - 2\Delta_{13}\,.\end{aligned}$$

The Γ's follow the same argument. Only B's are slightly different:

$$\begin{aligned}B_{41}' &= b_4' + b_1 \\ &= -b_4 + 3b_1 + 2b_2 + 2b_3 \\ &= 4b_1 + 2b_2 + 2b_3 - (b_1 + b_4) \\ &= -B_{41} + 2B_{12} + 2B_{13}\,.\end{aligned}$$

Thus the transformation is linear in the entries of Δ, Γ, and, B, with integer coefficients, as stated. □

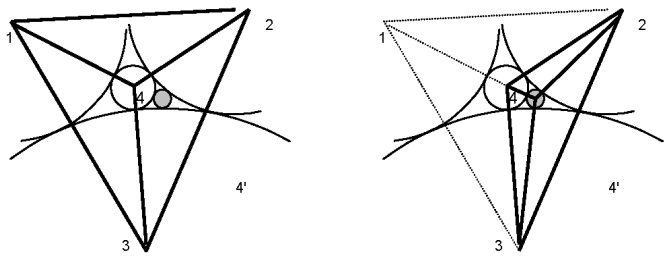


FIGURE 10. (a) A frame (b) and an adjacent frame. Circle $4'$ is shaded.

Actually, we have proved a little more:

Corollary 2. *If the (scaled) triangles formed by the initial Descartes configuration are integer, so are all of the triangles of the Apollonian window.*

Matrix description: Now we may rephrase our findings on transitions between frames in terms of matrices. The data for the Pythagorean triangles in each frame may be expressed as three vectors (columns), e.g., for the initial frame: $\Delta = [\Delta_{41}, \Delta_{42}, \Delta_{43}, \Delta_{12}, \Delta_{23}, \Delta_{31}]^T$ and similarly for Γ and B. Similarly for the second frame we have Δ', Γ' and B'.

Here are the three matrices of the transition from a frame to the corresponding subsequent frame:

$$A = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 1 & -2 & -2 \\ 2 & 0 & 1 & 0 & 2 & 0 \\ -2 & -1 & 0 & 0 & 0 & 2 \end{array}\right] \quad B = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & -2 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & -2 \\ 1 & 2 & 0 & 0 & 0 & 2 \end{array}\right]$$

$$C = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 2 & 2 & 0 & 0 \\ -1 & 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 & 2 & 1 \end{array}\right]$$

(vertical and horizontal lines in the matrices are drawn for easier inspection). Now acting with them on the initial vectors Δ and Γ one may reconstruct all the interior triangles. Vectors Δ and Γ may be combined into a single matrix, $T = [\Delta|\Gamma]$.

$$T = \begin{bmatrix} \Delta_{41} & \Gamma_{41} \\ \Delta_{42} & \Gamma_{42} \\ \Delta_{43} & \Gamma_{43} \\ \Delta_{12} & \Gamma_{12} \\ \Delta_{23} & \Gamma_{23} \\ \Delta_{31} & \Gamma_{31} \end{bmatrix} \quad B' = \begin{bmatrix} B_{41} \\ B_{42} \\ B_{43} \\ B_{12} \\ B_{23} \\ B_{31} \end{bmatrix}$$

The vector B is transformed to B' by matrices like A, B, C, where all of the entries are to be replaced by their absolute values.

We conclude with two examples of Apollonian packings in which every triangle for adjacent circles determines a Pythagorean triple, see Figures 10 and 11. Only some of the triangles are displayed.

6. Concluding remarks

Integral Apollonian disk packings have a topic of much interest for a while [2]-[4], [8]–[9] and their occurrence has been analyzed. Yet it is not clear whether the integral Apollonian disk packings admit coordinate systems in which the reduced coordinates are integral, except the known cases of Apollonian Strip and Apollonian Window. We have however a frequent occurrence of gaskets that admit integral Pythagorean triples, constructed in Sections 4 and 5.

Yet another intriguing remark concerns the master equation

$$B^2 + \mu^2 - nk = 0.$$

If we see it as $B^2 + \mu^2 =$ hypotenuse^2 – one may identify the corresponding (scaled) triangle in Figure 4 as the one with legs being the diameter $2/B$ of the encompassing circle and altitude h of the center of circle B_2 – the inversive image of circle B_2. Each solution to the master equation may also be viewed as an integral isotropic vector $[B, \mu, n, k]^T$ in the Minkowski space $\mathbb{R}^{3,1}$ *a la* Pedoe map from circles to vectors, see [6]. Such vectors represent points in this Minkowski space – degenerated circles of radius 0.

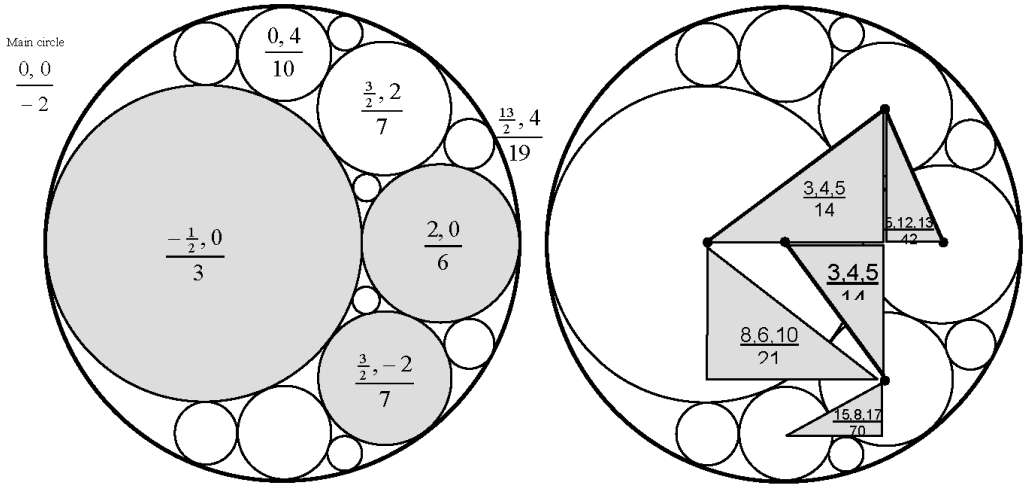


FIGURE 11. Arrangement $(b_0, b_1, b_2, b_3) = (-2, 3, 6, 7)$ or $(B, \mu, k, n) = (2, 4, 4, 5)$. Left side shows symbols of the principal and other circles, right side shows Pythagorean triples.

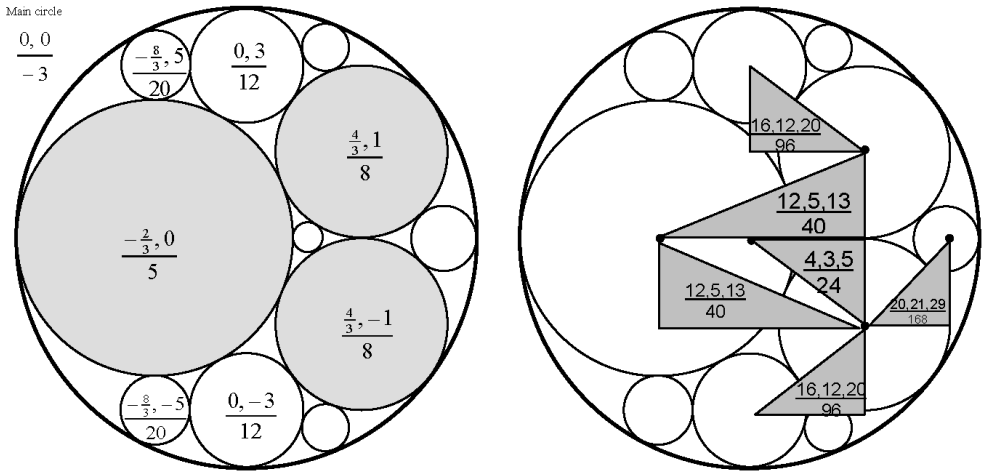


FIGURE 12. Arrangement $(b_0, b_1, b_2, b_3) = (-3, 5, 8, 8)$ or $(B, \mu, k, n) = (3, 4, 5, 5)$. Both symbols (left) and Pythagorean triples (right). The triangles are reported with absolute values of sides.

Appendix A. List of integral packings

Here we provide a list of the first 200 integer Apollonian packings. First, however, a few words on symmetry of these arrangements.

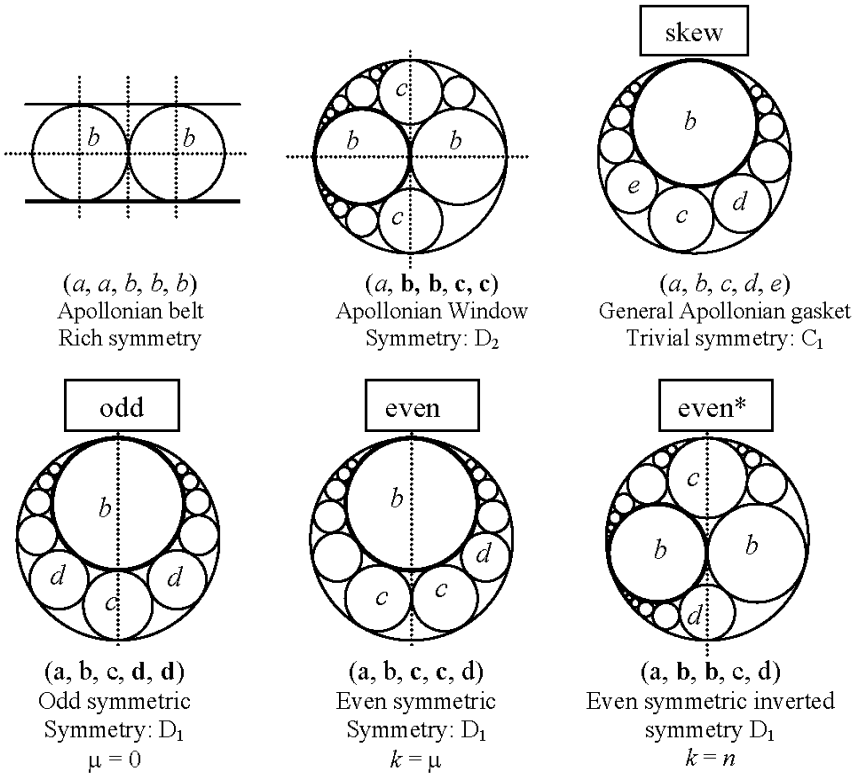


FIGURE A1. Types of symmetries and arrangements. The second circle, b, is thicker.

Only two integral Apollonian packings have rich symmetry: that of Apollonian Strip, and of Apollonian Window (D_2). Otherwise we have two cases, that of symmetry C_1 and D_1, which will be labeled as follows:

Skew means no nontrivial symmetry (thus symmetry of type C_1).

For cases with single axis symmetry (D_1) we shall distinguish two cases:

Odd – mirror symmetry D_1, with exactly three circles cut by the axis of symmetry

Even – mirror symmetry with infinitely many circles on the axis

Even* – even symmetry that is obtained "accidentally" by algorithm in Figure 3, where the two greatest congruent circles are not images of the congruent disks in the Apollonian strip

Consult Figure A1 for visualization.

Yet another attribute of an Apollonian packing is a **shift**, which we define as the degree the disk on the horizontal axis in the Apollonian strip is raised above this axis

in terms of the fraction of its radius.

$$\text{shift} = h/\rho = \frac{2\mu}{k}\,.$$

See Figure 4. It has values in the interval [0,1] and it measures the degree the gasket is off the axial symmetry: 0 for *odd* symmetry D_1 and 1 for *even* symmetry D_1. The fractional value indicates lack of axial symmetry, except the cases of "accidental" even* symmetry, when $n = k$ (this gives $B_1 = B_2$).

Table A1 displays first 24 integral Apollonian gaskets including symmetry type, principal disk quintet, the parameters B, k, n, μ, and the shift factor. The first even* case is also included.

Table A2 contains a longer list of integer Apollonian gaskets for the principal curvatures from 1 through 32. Curvature quintets (in brackets) are followed by shift factor.

		symmetry type	Curvatures -B0	B1	B2	B3	B4	Four indices B	k	n	mu	shift
1	1	odd	1	2	2	3	3	**1**	**1**	**1**	**0**	0
2	1	odd	2	3	6	7	7	**2**	**1**	**4**	**0**	0
3	1	odd	3	4	12	13	13	**3**	**1**	**9**	**0**	0
4	2	even	3	5	8	8	12	**3**	**2**	**5**	**1**	1
5	1	odd	4	5	20	21	21	**4**	**1**	**16**	**0**	0
6	2	even	4	8	9	9	17	**4**	**4**	**5**	**2**	1
7	1	odd	5	6	30	31	31	**5**	**1**	**25**	**0**	0
8	2	even	5	7	18	18	22	**5**	**2**	**13**	**1**	1
9	1	odd	6	7	42	43	43	**6**	**1**	**36**	**0**	0
10	2	odd	6	10	15	19	19	**6**	**4**	**9**	**0**	0
11	3	skew	6	11	14	15	23	**6**	**5**	**8**	**2**	4/5
12	1	odd	7	8	56	57	57	**7**	**1**	**49**	**0**	0
13	2	even	7	9	32	32	36	**7**	**2**	**25**	**1**	1
14	3	skew	7	12	17	20	24	**7**	**5**	**10**	**1**	2/5
15	1	odd	8	9	72	73	73	**8**	**1**	**64**	**0**	0
16	2	skew	8	13	21	24	28	**8**	**5**	**13**	**1**	2/5
17	3	even	8	12	25	25	33	**8**	**4**	**17**	**2**	1
18	1	odd	9	10	90	91	91	**9**	**1**	**81**	**0**	0
19	2	even	9	11	50	50	54	**9**	**2**	**41**	**1**	1
20	3	skew	9	14	26	27	35	**9**	**5**	**17**	**2**	4/5
21	4	skew	9	18	19	22	34	**9**	**9**	**10**	**3**	2/3
22	1	odd	10	11	110	111	111	**10**	**1**	**100**	**0**	0
23	2	odd	10	14	35	39	39	**10**	**4**	**25**	**0**	0
24	3	skew	10	18	23	27	35	**10**	**8**	**13**	**2**	1/2
...												
34	6	even*	12	25	25	28	48	**12**	**13**	**13**	**5**	10/13

TABLE A1. Bend quintets for Apollonian integer disk packings for principal curvatures 1 through 15, including four indices and shifts.

1 ... (0, 0, 1, 1, 1)

1 even (1, 2, 2, 3, 3) 0

1 odd (2, 3, 6, 7, 7) 0

1 odd (3, 4, 12, 13, 13) 0
2 even (3, 5, 8, 8, 12) 1

1 odd (4, 5, 20, 21, 21) 0
2 even (4, 8, 9, 9, 17) 1

1 odd (5, 6, 30, 31, 31) 0
2 even (5, 7, 18, 18, 22) 1

1 odd (6, 7, 42, 43, 43) 0
2 odd (6, 10, 15, 19, 19) 0
3 skew (6, 11, 14, 15, 23) 4/5

1 odd (7, 8, 56, 57, 57) 0
2 even (7, 9, 32, 32, 36) 1
3 skew (7, 12, 17, 20, 24) 2/5

1 odd (8, 9, 72, 73, 73) 0
2 skew (8, 13, 21, 24, 28) 2/5
3 even (8, 12, 25, 25, 33) 1

1 odd (9, 10, 90, 91, 91) 0
2 even (9, 11, 50, 50, 54) 1
3 skew (9, 14, 26, 27, 35) 4/5
4 skew (9, 18, 19, 22, 34) 2/3

1 odd (10, 11, 110, 111, 111) 0
2 odd (10, 14, 35, 39, 39) 0
3 skew (10, 18, 23, 27, 35) 1/2

1 odd (11, 12, 132, 133, 133) 0
2 even (11, 13, 72, 72, 76) 1
3 skew (11, 16, 36, 37, 45) 4/5
4 skew (11, 21, 24, 28, 40) 2/3

1 odd (12, 13, 156, 157, 157) 0
2 odd (12, 21, 28, 37, 37) 0
3 skew (12, 17, 41, 44, 48) 2/5
4 even (12, 16, 49, 49, 57) 1
5 skew (12, 21, 29, 32, 44) 2/3
6 skew (12, 25, 25, 28, 48) 10/13

1 odd (13, 14, 182, 183, 183) 0
2 even (13, 15, 98, 98, 102) 1
3 skew (13, 18, 47, 50, 54) 2/5
4 skew (13, 23, 30, 38, 42) 1/5

1 odd (14, 15, 210, 211, 211) 0
2 odd (14, 18, 63, 67, 67) 0
3 skew (14, 19, 54, 55, 63) 4/5
4 skew (14, 22, 39, 43, 51) 1/2
5 skew (14, 27, 31, 34, 54) 10/13

1 odd (15, 16, 240, 241, 241) 0
2 odd (15, 24, 40, 49, 49) 0
3 even (15, 17, 128, 128, 132) 1
4 skew (15, 24, 41, 44, 56) 2/3
5 skew (15, 28, 33, 40, 52) 6/13
6 skew (15, 32, 32, 33, 65) 16/17
1 odd (16, 17, 272, 273, 273) 0
2 even (16, 20, 81, 81, 89) 1
3 skew (16, 21, 68, 69, 77) 4/5
4 skew (16, 29, 36, 45, 53) 4/13
5 skew (16, 32, 33, 41, 57) 1/2

1 odd (17, 18, 306, 307, 307) 0
2 even (17, 19, 162, 162, 166) 1
3 skew (17, 22, 75, 78, 82) 2/5
4 skew (17, 27, 46, 54, 58) 1/5
5 skew (17, 30, 42, 43, 67) 12/13

1 odd (18, 19, 342, 343, 343) 0
2 odd (18, 22, 99, 103, 103) 0
3 skew (18, 23, 83, 86, 90) 2/5
4 skew (18, 31, 43, 54, 58) 2/13
5 skew (18, 26, 59, 63, 71) 1/2
6 skew (18, 27, 55, 58, 70) 2/3
7 skew (18, 35, 38, 47, 63) 8/17

1 odd (19, 20, 380, 381, 381) 0
2 even (19, 21, 200, 200, 204) 1
3 skew (19, 24, 92, 93, 101) 4/5
4 skew (19, 29, 56, 60, 72) 2/5
5 skew (19, 32, 48, 53, 69) 8/13
6 skew (19, 36, 44, 45, 77) 16/17

1 odd (20, 21, 420, 421, 421) 0
2 odd (20, 36, 45, 61, 61) 0
3 even (20, 24, 121, 121, 129) 1
4 skew (20, 33, 52, 57, 73) 8/13
5 skew (20, 37, 45, 52, 72) 10/17
6 even (20, 36, 49, 49, 81) 1

1 odd (21, 22, 462, 463, 463) 0
2 odd (21, 30, 70, 79, 79) 0
3 even (21, 23, 242, 242, 246) 1
4 skew (21, 34, 55, 66, 70) 2/13
5 skew (21, 38, 47, 62, 66) 2/17
6 skew (21, 26, 110, 111, 119) 4/5
7 skew (21, 30, 71, 74, 86) 2/3
8 skew (21, 31, 66, 70, 82) 3/5
9 skew (21, 39, 46, 58, 70) 1/3
10 even (21, 39, 50, 50, 86) 1

1 odd (22, 23, 506, 507, 507) 0
2 odd (22, 26, 143, 147, 147) 0
3 skew (22, 27, 119, 122, 126) 2/5
4 skew (22, 30, 83, 87, 95) 1/2
5 skew (22, 39, 51, 62, 74) 6/17
6 skew (22, 42, 47, 59, 75) 2/5
7 skew (22, 35, 62, 63, 87) 12/13

1 odd (23, 24, 552, 553, 553) 0
2 even (23, 25, 288, 288, 292) 1
3 skew (23, 28, 129, 132, 136) 2/5
4 skew (23, 33, 76, 84, 88) 1/5
5 skew (23, 36, 64, 73, 81) 4/13
6 skew (23, 40, 57, 60, 88) 14/17
7 skew (23, 48, 49, 52, 96) 22/25

1 odd (24, 25, 600, 601, 601) 0
2 odd (24, 33, 88, 97, 97) 0
3 even (24, 28, 169, 169, 177) 1
4 skew (24, 29, 140, 141, 149) 4/5
5 skew (24, 44, 53, 69, 77) 1/5
6 skew (24, 33, 89, 92, 104) 2/3
7 skew (24, 37, 69, 76, 88) 6/13
8 skew (24, 40, 61, 69, 85) 1/2
9 skew (24, 41, 60, 65, 89) 12/17
10 skew (24, 49, 49, 60, 88) 14/25

1 odd (25, 26, 650, 651, 651) 0
2 even (25, 27, 338, 338, 342) 1
3 skew (25, 42, 62, 75, 83) 4/17
4 skew (25, 38, 75, 78, 98) 10/13
5 skew (25, 50, 51, 66, 86) 2/5
6 skew (25, 50, 54, 59, 99) 4/5

1 odd (26, 27, 702, 703, 703) 0
2 odd (26, 30, 195, 199, 199) 0
3 skew (26, 31, 162, 163, 171) 4/5
4 skew (26, 34, 111, 115, 123) 1/2
5 skew (26, 43, 66, 79, 87) 4/17
6 skew (26, 51, 55, 66, 94) 14/25
7 skew (26, 46, 63, 67, 99) 4/5

1 odd (27, 28, 756, 757, 757) 0
2 even (27, 29, 392, 392, 396) 1
3 skew (27, 32, 173, 176, 180) 2/5
4 skew (27, 37, 100, 108, 112) 1/5
5 skew (27, 36, 109, 112, 124) 2/3
6 skew (27, 45, 68, 80, 92) 1/3
7 skew (27, 40, 85, 88, 108) 10/13
8 skew (27, 53, 56, 72, 92) 5/13
9 skew (27, 44, 72, 77, 101) 12/17
10 skew (27, 52, 61, 64, 108) 22/25

1 odd (28, 29, 812, 813, 813) 0
2 odd (28, 44, 77, 93, 93) 0
3 skew (28, 33, 185, 188, 192) 2/5
4 even (28, 32, 225, 225, 233) 1
5 skew (28, 41, 89, 96, 108) 6/13
6 skew (28, 53, 60, 77, 93) 8/25
7 skew (28, 48, 69, 77, 101) 3/5
8 skew (28, 45, 77, 80, 108) 14/17
9 even (28, 44, 81, 81, 113) 1
10 skew (28, 57, 60, 65, 113) 24/29

1 odd (29, 30, 870, 871, 871) 0
2 even (29, 31, 450, 450, 454) 1
3 skew (29, 34, 198, 199, 207) 4/5
4 skew (29, 42, 94, 103, 111) 4/13
5 skew (29, 39, 114, 118, 130) 2/5
6 skew (29, 46, 79, 90, 102) 6/17
7 skew (29, 54, 63, 82, 94) 6/25
8 skew (29, 55, 66, 70, 114) 11/13

1 odd (30, 31, 930, 931, 931) 0
2 odd (30, 34, 255, 259, 259) 0
3 odd (30, 39, 130, 139, 139) 0
4 odd (30, 55, 66, 91, 91) 0
5 skew (30, 47, 83, 98, 102) 2/17
6 skew (30, 38, 143, 147, 155) 1/2
7 skew (30, 39, 131, 134, 146) 2/3
8 skew (30, 55, 67, 82, 102) 2/5
9 skew (30, 43, 102, 103, 127) 12/13
10 skew (30, 59, 66, 71, 119) 24/29

1 odd (31, 32, 992, 993, 993) 0
2 even (31, 33, 512, 512, 516) 1
3 skew (31, 44, 105, 116, 120) 2/13
4 skew (31, 57, 68, 92, 96) 1/13
5 skew (31, 36, 224, 225, 233) 4/5
6 skew (31, 41, 128, 132, 144) 2/5
7 skew, 31, 48, 89, 96) ft= 10/17
8 skew (31, 60, 65, 84, 104) 10/29
9 skew (31, 56, 72, 81, 113) 16/25

1 odd (32, 33, 1056, 1057, 1057) 0
2 skew (32, 37, 237, 240, 244) 2/5
3 skew (32, 57, 73, 96, 100) 2/25
4 even (32, 36, 289, 289, 297) 1
5 skew (32, 45, 112, 117, 133) 8/13
6 skew (32, 48, 97, 105, 121) 1/2
7 skew (32, 52, 85, 93, 117) 2/3
8 skew (32, 61, 69, 84, 112) 14/29
9 skew (32, 49, 96, 97, 129) 16/17

TABLE A2. Bend quintets for Apollonian integer disk packings for principal curvatures 1 through 32.

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